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EXTERIOR BALLISTICS FOR AIRBORNE APPLICATIONS

John M. Norwood

Applied Research Laboratories
The University of Texas at Austin

TECHNICAL REPORT AFAL-TR-73-196

June 1973

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EXTERIOR BALLISTICS FOR AIRBORNE APPLICATIONS

John M. Norwood

FOREWORD

This technical report is submitted in accordance with the requirements of Contract F33615-70-C-1162, Exhibit B, Sequence No. B002. The work documented herein was accomplished under Project 7629, Task 03 during the periods of July 1971 to August 1971 and January 1973 to April 1973 under the cognizance of Mr. Leo Krautmann, Project Engineer, AFAL/NVT-1, Air Force Avionics Laboratory, Wright-Patterson Air Force Base, Ohio.

Portions of this report have been documented elsewhere by the Aerospace Technology Division, Applied Research Laboratories (ARL), The University of Texas at Austin, Austin, Texas, as ARL reports, and other portions dealing with the fundamentals of ballistics are based upon material found in the References. This report provides the Air Force and Industry with a comprehensive treatment of exterior ballistics, as applied to airborne applications, under one cover.

This report was submitted by the author April 1973 and is assigned the originator's report number UT/ARL-TR-73-15.

This report has been reviewed and is approved for publication.

Warren M. Hansen

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Acting Deputy Director
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ABSTRACT

Methods of exterior ballistics applicable to utilization in modern airborne fire control system design are documented herein. The fundamentals of exterior ballistics are included, along with a description of currently used ballistic and aerodynamic notations, a discussion of the limitations of the semi-empirical aerodynamic force and moment system, and methods of preparing aerodynamic data for use in trajectory computation. Tutorial material is provided to give the reader an understanding of windage jump caused by the complicated angular motion of a spinning projectile. The Siacci method is described and means for improving its accuracy are developed. Six-degree-of-freedom equations are derived in several different formulations for exploratory studies and digital computer computations, and the development of approximate equations for rapid evaluation of trajectory tables is included. Methods for calculating trajectory initial conditions are provided for shells fired from a turreted, gatling gun in a maneuvering aircraft, and the problems of ballistic and kinematic prediction are discussed briefly. The material covered herein should provide personnel in the Air Force and in industry with sufficient knowledge of exterior ballistics for advanced fire control system design.

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LIST OF SYMBOLS

a_o	Ratio of local speed of sound to sea level speed of sound
A	Axial moment of inertia
A(U)	Siacci altitude function
B	Transverse moment of inertia
c	$= c' + \frac{c''}{s_o - 1}$
c'	$= \frac{\rho_o d^2}{2m} \left[K_L + \frac{md^2}{B} K_H \right]$
c"	$= \frac{\rho_o d^2}{2m} K_D$
C	Ballistic coefficient
C_D	Drag coefficient
C_ℓ	Damping-in-spin coefficient
$C_{\ell p}$	$\frac{\partial C_\ell}{\partial (pd/2V)}$
C_m	Pitching-moment coefficient
$C_{m\alpha}$	$= \frac{\partial C_m}{\partial \alpha}$
$C_{m\dot{\alpha}}$	$= \frac{\partial C_m}{\partial (\dot{\alpha} d/2V)}$
$C_{mp\beta}$	$= C_{Mp\alpha}$
$C_{mp\dot{\beta}}$	$= C_{Mp\dot{\alpha}}$

LIST OF SYMBOLS (continued)

C_{mpr} $= C_{Mpq}$

$C_{mp\dot{r}}$ $= C_{Mp\dot{q}}$

C_{mq} $= \dfrac{\partial C_m}{\partial (qd/2V)}$

$C_{m\dot{q}}$ $= \dfrac{\partial C_m}{\partial (\dot{q}d^2/4V^2)}$

$C_{M\alpha}$ $= C_{m\alpha}$

$C_{M\dot{\alpha}}$ $= C_{m\dot{\alpha}}$

$C_{Mp\alpha}$ $= \dfrac{\partial^2 C_m}{\partial (pd/2V)\, \partial \alpha}$

$C_{Mp\dot{\alpha}}$ $= \dfrac{\partial^2 C_m}{\partial (pd/2V)\, \partial (\dot{\alpha}d/2V)}$

C_{Mpq} $= \dfrac{\partial^2 C_m}{\partial (pd/2V)\, \partial (qd/2V)}$

$C_{Mp\dot{q}}$ $= \dfrac{\partial^2 C_m}{\partial (pd/2V)\, \partial (\dot{q}d^2/4V^2)}$

C_{Mq} $= C_{mq}$

$C_{M\dot{q}}$ $= C_{m\dot{q}}$

C_n Yawing-moment coefficient

LIST OF SYMBOLS (continued)

$C_{n\beta}$	$= -C_{m\alpha}$
$C_{n\dot{\beta}}$	$= -C_{m\dot{\alpha}}$
$C_{np\alpha}$	$= C_{Mp\alpha}$
$C_{np\dot{\alpha}}$	$= C_{Mp\dot{\alpha}}$
C_{npq}	$= -C_{Mpq}$
$C_{np\dot{q}}$	$= -C_{Mp\dot{q}}$
C_{nr}	$= C_{mq}$
$C_{n\dot{r}}$	$= C_{m\dot{q}}$
C_N	Normal-force coefficient
$C_{N\alpha}$	$\frac{\partial C_N}{\partial \alpha}$
$C_{N\dot{\alpha}}$	$\frac{\partial C_N}{\partial (\dot{\alpha}d/2V)}$
$C_{Np\alpha}$	$= \frac{\partial^2 C_N}{\partial (pd/2V)\, \partial \alpha}$
$C_{Np\dot{\alpha}}$	$= \frac{\partial^2 C_N}{\partial (pd/2V)\, \partial (\dot{\alpha}d/2V)}$
C_{Npq}	$= \frac{\partial^2 C_N}{\partial (pd/2V)\, \partial (qd/2V)}$

LIST OF SYMBOLS (continued)

$C_{Np\dot{q}}$	$=$	$\dfrac{\partial^2 C_N}{\partial(pd/2V)\,\partial(\dot{q}d^2/4V^2)}$
C_{Nq}	$=$	$\dfrac{\partial C_N}{\partial(qd/2V)}$
$C_{N\dot{q}}$	$=$	$\dfrac{\partial C_N}{\partial(\dot{q}d^2/4V^2)}$
C_Y		Side-force coefficient
$C_{Y\beta}$	$=$	$C_{N\alpha}$
$C_{Y\dot{\beta}}$	$=$	$C_{N\dot{\alpha}}$
$C_{Yp\alpha}$	$=$	$-C_{Np\alpha}$
$C_{Yp\dot{\alpha}}$	$=$	$-C_{Np\dot{\alpha}}$
C_{Ypq}	$=$	C_{Npq}
$C_{Yp\dot{q}}$	$=$	$C_{Np\dot{q}}$
C_{Yr}	$=$	$-C_{Nq}$
$C_{Y\dot{r}}$	$=$	$-C_{N\dot{q}}$

LIST OF SYMBOLS (continued)

C_Z	$= C_N$
$C_{Z\alpha}$	$= C_{N\alpha}$
$C_{Z\dot{\alpha}}$	$= C_{N\dot{\alpha}}$
$C_{Zp\beta}$	$= C_{Np\alpha}$
$C_{Zp\dot{\beta}}$	$= C_{Np\dot{\alpha}}$
C_{Zpr}	$= C_{Npq}$
$C_{Zp\dot{r}}$	$= C_{Np\dot{q}}$
C_{Zq}	$= C_{Nq}$
$C_{Z\dot{q}}$	$= C_{N\dot{q}}$
d	Projectile diameter
f	$= F_2 + iF_3$
$\vec{F}$	Force
$\vec{g}$	Acceleration due to gravity
g	$= G_2 + iG_3$ (only in Section II)
$\vec{G}$	Torque
$\vec{H}$	Angular momentum
I(U)	Siacci inclination function

LIST OF SYMBOLS (continued)

$\vec{J}$	Windage jump
K_A	Spin-deceleration coefficient
K_D	Drag coefficient
K_{DA}	Axial drag coefficient
K_{Do}	Zero-yaw drag
$K_{D_{\delta^2}}$	Yaw drag coefficient
K_F	Magnus force coefficient
K_H	Damping force coefficient
K_L	Lift force coefficient
K_M	Overturning moment coefficient
K_N	Normal force coefficient
K_T	Magnus moment coefficient
m	Projectile mass
M	Mach number
p	Roll rate of projectile in Section II; roll rate of aircraft in Section VI
$\vec{P}$	Siacci range

LIST OF SYMBOLS (continued)

q	Pitch rate of projectile in Section II; pitch rate of aircraft in Section VI
$\vec{Q}$	Siacci gravity drop
r	Yaw rate of projectile in Section II; pitch rate of aircraft in Section VI
R_e	Reynolds number
s	Arc length along a trajectory
$\vec{s}$	Unit vector along $\vec{H}$ in Section IV
s_o	Dynamic stability factor
S	Projectile reference area (only in Subsection 9 of Section II)
$\vec{S}$	Swerve
$S(U)$	Siacci space function
t	Time
$T(U)$	Siacci time function
u	$= \dot{P}$, the pseudovelocity, in Section V; projectile X velocity in Section II, Subsection 9; elsewhere $u = \lvert \vec{u} \rvert$
$\vec{u}$	Projectile velocity
$\vec{u}_o$	Initial projectile velocity
U	$= u/a_o$
v	Projectile Y velocity
$\vec{V}$	Projectile velocity
$\vec{V}_A$	Aircraft velocity

LIST OF SYMBOLS (continued)

$\vec{V}_M$	Projectile muzzle velocity
V_s	Speed of sound
w	Projectile Z velocity
α	Azimuth angle of $\vec{u}$; angle of attack (aerodynamic notation) used only in Section II, Subsections 9, 10, and 11
β	Yaw angle (aerodynamic notation)
δ	Yaw angle (ballistic notation)
δ_o	Initial yaw angle
θ	Elevation angle of $\vec{u}$ (Fig. 10) except in Section II where θ is the angle of symmetry, and in Section III where θ is the angle between the spin axis and the x axis (Fig. 5)
θ_o	Initial value of θ
λ	$= \dfrac{u_2 + iu_3}{u}$, cross velocity
μ	$= \dfrac{(\omega_2 + i\omega_3)d}{u}$, cross angular velocity
μ_a	Viscosity of the air
ν	Dimensionless spin
ρ	Air density
ρ_o	Air density at sea level
σ	$= \rho/\rho_o$

LIST OF SYMBOLS (concluded)

ϕ	Precession angle defined in Fig. 5 for Section III, and in Fig. 11 for all other applications
ϕ_o	Initial value of ϕ
ϕ'	$= \phi - \phi_o$
$\vec{\omega}$	Angular velocity of shell
$\vec{\Omega}$	Angular velocity of rotating coordinate system

SECTION I

INTRODUCTION

For the design of a modern airborne gun fire control system, a knowledge of exterior ballistics is a necessity. The complicated motion of a spinning projectile, fired crosswind from a gun in a high-speed aircraft, and the accurate prediction of such motion requires careful study.

These statements are borne out by consideration of the magnitudes of aerodynamic forces and moments acting upon a projectile in flight. It is almost incredible that the aerodynamic drag on the 20-mm, M56 round fired at sea level at 4000 ft/sec is about 20 lb, and that the left force at right angles to the velocity vector is about 9 lb if the angle of attack of the projectile is 3 deg. The 20-mm round itself weighs only 0.22 lb. The effect of the lift, and other smaller forces, is to deflect the motion of the projectile away from its initial direction of motion, and, as a rule of thumb, this deflection, called the windage jump, is one milliradian (mr) per initial degree of angle of attack. In the example above, the windage jump is 3 mr and one might ask how it is that a 9-lb force acting at right angles to the velocity vector of a 0.22-lb projectile produces only a 3-mil deflection. The answer is that the lift force precesses with the projectile about the velocity vector and nearly cancels out.

That a careful study of the six-degree-of-freedom motion of a projectile is necessary should now be apparent. The deviations from straight-line motion of a spinning projectile in flight are due to complicated interactions of aerodynamic forces and moments, gravity, and gyroscopic effects. For accurate prediction of projectile motion, these effects must be taken into account.

This report is limited to an investigation of the motion of a spinning projectile fired from a rifle. Finned missiles and rockets are not included.

Whereas accurate six-degree-of-freedom projectile trajectories can be calculated directly from Newton's laws of motion, computations of this type are currently not believed feasible in airborne fire control applications because excessive computer time is required. Careful study, however, leads to the selection of appropriate algorithms which yield trajectory data within the error budget alloted to exterior ballistics calculations, and which can be utilized in airborne applications.

Because of the 9 or 10-year lapse of interest by the Air Force in exterior ballistics, it is felt that a presentation of the fundamentals of the subject is not out of place in this report. New personnel in the Air Force and in industry are not likely to be knowledgeable in this subject, nor is the necessary literature readily available. The best available book on exterior ballistics is the one by McShane, Kelley, and Reno (Ref. 1), but this book is not well suited to current needs. Other useful information is to be found in reports written mostly in the 1950's and earlier which may be hard to obtain, such as in Refs. 2 through 20. Most of this work is based upon the early work of Fowler, Gallop, Lock, and Richmond (Ref. 21), and upon the works of Nielsen and Synge (Ref. 22), and of Maple and Synge (Ref. 23). A useful book on fire control is NAVORD Report 1493 (Ref. 24), and the book by Davis, Follin, and Blitzer (Ref. 25) provides information applicable to the exterior ballistics of shells. The notation of Ref. 25 is somewhat different than used elsewhere, but the book provides very good insight into projectile motion. After burning, the motion of a spin-stabilized rocket is the same as that of a shell. Two other books on exterior ballistics are the ones by Moulton (Ref. 26) and Bliss (Ref. 27).

The aerodynamic force and moment system in current use is semi empirical. Whereas the functional form of the forces and moments may be partially derived from considerations of dimensional analysis and projectile symmetry, aerodynamic theory has failed to produce accurate expressions for these forces and moments. Aerodynamic data used in trajectory computation is acquired from wind tunnel measurements and

free-flight tests. The arguments of dimensional analysis and symmetry which are used to develop the aerodynamic force and moment system in use are presented in Section II. Discussion of the measurement of aerodynamic data is excluded. Also included in Section II is a comparison of the ballistic K notation and the aerodynamic C notation, physical identification of the various aerodynamic forces and moments, a discussion of weaknesses in the force and moment system, and modeling of aerodynamic data for computer utilization.

Section III is tutorial and should provide the reader with the necessary background to understand the nature of the angular motion of a spinning projectile. An understanding of the fundamental causes of windage jump follows from knowledge of projectile angular motion.

Equations of motion suitable for computation of trajectories on a computer are given in Section IV. Included are a matrix formulation and an Euler angle development of the six-degree-of-freedom equations. Also included is a set of approximate equations which are useful for rapid generation of trajectory data when computer time is a factor.

The basic Siacci method is presented in Section V, along with methods to improve its accuracy. The Siacci method is a good candidate for onboard utilization because it provides an approximate closed form solution to the trajectory computation problem.

Equations for initial conditions for trajectory calculations are derived in Section VI. These equations apply to a turreted, gatling gun fired from a moving aircraft. They are possibly more complicated than necessary, but they may be simplified as any application permits. A brief discussion of the problems imposed by kinematic prediction is also included in this section.

SECTION II

AERODYNAMIC FORCES AND MOMENTS

1. General

The aerodynamic forces and moments acting on a projectile in flight result from frictional forces and pressure distributions over the projectile body which in turn result from the motion of the projectile through the air mass. A complete and accurate solution for projectile motion would therefore involve solution of the equations of fluid flow around the projectile. In practical applications, this is beyond the capabilities of any available digital computer, but fortunately this is not necessary. The alternative is a semi-empirical approach whereby aerodynamic forces and moments are measured in wind tunnels and by means of free-flight tests. This aerodynamic data is modeled in a form suitable for computer use and employed in trajectory calculations. The means by which this data is modeled are treated in this section.

Dimensional analysis is a study of certain mathematical relationships, explainable in terms of the dimensionality of measurements, which exist between physically measurable quantities. Physically measurable quantities have dimensions such as mass (M), length (L), Time (T), or velocity (LT^{-1}), and physical laws are independent of these units of measure. Certain restrictions are imposed by nature upon the functional form of mathematical relations describing such laws, and dimensional analysis identifies these restrictions. In the semi-empirical development of the aerodynamic force and moment system applicable to a spinning shell, arguments based upon dimensional analysis and symmetry play an important role.

A shell possesses two types of symmetry, namely rotational and mirror. An object has rotational symmetry if "it looks the same" when it has been rotated through a given angle about a particular axis. Similarly, an object has mirror symmetry if it looks the same under certain conditions when viewed in a mirror. Dimensional analysis and

symmetry are used to derive the functional form of the aerodynamic force and moment system in terms of the aerodynamic or ballistic coefficients.

Since the approach used to mathematically describe aerodynamic forces and moments is semi-empirical, it has certain weaknesses. These weaknesses are discussed briefly.

The approach herein is to first mathematically model the force and moment system and then to identify the various forces and moments. One might expect the analysis to proceed in the other direction, but use of this systematic approach uncovers unsuspected forces and moments. The systematic approach should not trouble the reader since he will already be aware of the existence of certain forces such as the drag, the lift, the Magnus force, the Magnus moment, etc., and their appearance in the equations to be developed will be no surprise.

This section also contains a comparison of the different notations which are in current use for the description of the aerodynamic forces and moments which act upon a projectile in free flight. The terminology used in the literature to describe the aerodynamics of the operational 20-mm, M56 round is the K, or ballistic, notation, whereas that used to describe new 20-mm, 25-mm, and 30-mm rounds under development is the C, or aerodynamic, notation. Much of the old ballistics literature which employs the K notation is still of interest, but the new C notation is used by wind tunnel and free-flight test range personnel in the collection of new ballistics data. Familiarity with both notations is mandatory.

Choice of an appropriate notation for ballistics is subjective. The C notation is somewhat awkward in use in mathematical developments, although it does exhibit a lucidness of meaning which is absent in the K notation. The two systems of notation are mathematically equivalent, and the K notation is used throughout this report, since most of the literature upon which this report is based is written in terms of the K notation.

2. Dimensional Analysis

The subject of dimensional analysis is an important part of the basic theory of aerodynamics, and since it is well documented (see, for example, Refs. 1, 28, and 29), it will not be the purpose of this section to present a detailed account of the subject. Rather, results derived from the theory will be applied to the problem at hand.

If α, β, ..., ω are physically measurable quantities (with dimensions) which satisfy an equation which represents a physical law, such as

$$\phi(\alpha, \beta, \ldots, \omega) = 0$$

this relation should also be satisfied if the units of measure of $\alpha, \beta, \ldots, \omega$ are changed (e.g., from grams, centimeters, and seconds to pounds, feet, and hours). This is reasonable, since one would not expect the functional form of ϕ to change under such a transformation. A theorem of dimensional analysis, known as the Buckingham Π - theorem, follows from this assertion. Crudely put, the theorem states that the relation $\phi(\alpha, \beta, \ldots, \omega) = 0$ can be replaced by another dimensionless relation

$$\psi\left(\pi_1, \pi_2, \ldots, \pi_{n-m}\right) = 0$$

where n is the number of parameters in the set of measurable quantities α, β, ..., ω, and m is the number of fundamental dimensions (pounds, feet, seconds, etc.). The π_i, for i = 1, 2, ..., n-m, are dimensionless products of the measurable quantities of the form

$$\pi_i = \alpha^{a_i} \beta^{b_i} \ldots \omega^{z_i}$$

One would expect intuitively the existence of a function such as ψ. The nature of a physical law is independent of the units of measure by which it is described, and a function such as ψ should be unaffected by

changes in units. Since the products π_i (for i = 1, 2, ..., n-m, are dimensionless, they do not change value with changes in units and it follows that ψ is unaffected by such changes. The function ϕ is essentially a rearrangement of ψ.

As an example of the use of the Buckingham Π-theorem, consider the motion of a body with acceleration, a, and initial velocity, v. The body is observed to move a distance, s, in time, t, and this motion is described, of course, by the formula

$$\phi(s, v, a, t) = s - vt - \frac{1}{2} at^2 = 0$$

But suppose, for the sake of argument, that this formula is not known, and that the object of an experiment is to discover the functional form of ϕ. This can be done by varying v and a in some systematic manner and by measuring s as a function of t. Now, there are four measurable quantities, s, v, a, and t, and two fundamental dimensions, length, L, and time, T, and, according to the Buckingham Π-theorem, there are two dimensionless products, π_1 and π_2. These are

$$\pi_1 = \frac{sa}{v^2}$$

$$\pi_2 = \frac{at}{v}$$

and there must be a dimensionless function

$$\psi\left(\frac{sa}{v^2}, \frac{at}{v}\right) = 0$$

which describes the motion of the body in question. It is obvious that a function of two parameters is simpler than a function of four parameters, and so it is easier to fit ψ to the data than to fit ϕ. It is found that

$$\psi\left(\frac{sa}{v^2}, \frac{at}{v}\right) = \frac{sa}{v^2} - \frac{at}{v} - \frac{1}{2}\left(\frac{at}{v}\right)^2 = 0$$

That ψ is equivalent to ϕ may be proven by multiplying ϕ through by a/v^2. In general, The Buckingham Π-theorem is used to reduce the number of independent parameters and thereby simplify formulae representing physical laws.

In the present application, the measurable quantities which describe the motion of a projectile through the air and their units are as follows:

Components of projectile velocity	$\vec{u}$	L/T
Components of projectile angular velocity	$\vec{\omega}$	T^{-1}
Air density	ρ	M/L^3
Viscosity of the air	μ_a	M/LT
Speed of sound in air	V_s	L/T
Projectile diameter	d	L
Force on the projectile	F	ML/T^2

There is a total of $n = 11$ measurable quantities (three each for $\vec{u}$ and $\vec{\omega}$), a total of $m = 3$ fundamental dimensions (M, L, and T), and it follows that there should be $n - m = 8$ unique dimensionless products. With u instead of u_1, they are as follows:

$\dfrac{F}{\rho d^2 u^2}$	Dimensionless force
$M = \dfrac{u}{V_s}$	Mach number
$R_e = \dfrac{\rho u d}{\mu_a}$	Reynolds number

$\nu = \frac{\omega_1 d}{u}$ Dimensionless spin

$\frac{u_2}{u}$ and $\frac{u_3}{u}$ Components of cross velocity

$\frac{\omega_2 d}{u}$ and $\frac{\omega_3 d}{u}$ Components of cross angular velocity

Thus, there is a dimensionless relation which is a function of these dimensionless products. If this relation is solved for $F/\rho d^2 u^2$, an equation of the form

$$F = \rho d^2 u^2 K \left(M, R_e, \nu, \frac{u_2}{u}, \frac{u_3}{u}, \frac{\omega_2 d}{u}, \frac{\omega_3 d}{u}\right)$$

is obtained in which the function K is dimensionless.

Although it was stated that the components of $\vec{u}$ are among the measurable quantities, u is used herein instead of u_1. The principal references for this development are Refs. 1, 3, and 10. In Refs. 1 and 10, u_1 is used, whereas u is used in Ref. 3. The arguments of dimensional analysis are in no way affected by this choice, and while the use of u_1 is advantageous for some purposes, the use of u is more common.

This development ignores the fact that the aerodynamic force can be resolved in three orthogonal directions. There are actually three relations such as the one above, and there are also three equations for the aerodynamic moment of the form

$$G = \rho d^3 u^2 K' \left(M, R_e, \nu, \frac{u_2}{u}, \frac{u_3}{u}, \frac{\omega_2 d}{u}, \frac{\omega_3 d}{u}\right)$$

in which K and K' are known as aerodynamic or ballistic coefficients. It will be shown in the following sections that the components of $\vec{F}$ and $\vec{G}$ can be represented as sums of expressions similar to those above containing dimensionless functions which are also known as aerodynamic

coefficients. This will follow from arguments applicable to projectile symmetry.

3. Rotational Symmetry

In the preceding subsection, dimensional analysis was employed to ascertain information as to the nature of the functional form of the aerodynamic forces and moments which act upon a projectile in free flight. In this subsection and the next, more information of the same type will be obtained by examination of the properties of symmetry of a projectile. Rotational symmetry will be treated in this subsection and mirror symmetry in the next (Refs. 1, 3, 8, 10, 22, and 23).

In vague terms, a projectile is described as being round. What is meant, of course, is that the appearance of the projectile is unchanged by a rotation about its longitudinal axis. It follows under rather general assumptions as to surface finish, etc. that aerodynamic forces and moments are also unchanged by such a rotation. A projectile with these properties is said to possess rotational symmetry and these properties may be exploited to gain additional information as to the nature of the aerodynamic forces and moments.

It is assumed that the projectile in question has an angle, θ, of rotational symmetry, where $0 < \theta < \pi$, and that aerodynamic forces and moments are unaffected by rotating the projectile through the angle θ. It is noted in passing that the analysis to follow is applicable not only to shell, but to missiles with three or more symmetrically located fins.

It is necessary to define a coordinate system and parameters needed in the development. Aerodynamic forces and moments are defined with respect to a right-handed, rectangular, missile-fixed x_1, x_2, x_3 coordinate system with the origin at the center of mass of the projectile, and with the x_1 axis directed toward the nose along the longitudinal axis. The x_2 and x_3 axis directions are fixed in the projectile. Projectile velocity, angular velocity, aerodynamic force, and aerodynamic moment are represented by symbols $\vec{u}$, $\vec{\omega}$, $\vec{F}$, and $\vec{G}$, respectively, and components of these vectors along a designated axis are

denoted by the subscript denoting that axis (e.g., u_1 is the component of $\vec{u}$ along x_1).

In the development of expressions for aerodynamic forces and moments, it is convenient to use a complex number notation. The complex number notation provides a convenient way of designating directions of aerodynamic forces and moments, and it is useful in exploiting rotational and mirror symmetry. The x_2, x_3 plane is taken to be the complex plane, with x_3 the imaginary axis. Let

$$\lambda = \frac{u_2 + i u_3}{u} \qquad \text{(Cross velocity)} \qquad (1)$$

$$\mu = \frac{(\omega_2 + i\omega_3) d}{u} \qquad \text{(Cross angular velocity)} \qquad (2)$$

$$f = F_2 + i F_3 \qquad \text{(Cross force)} \qquad (3)$$

$$g = G_2 + i G_3 \qquad \text{(Cross moment)} \qquad (4)$$

From the results of the preceding section, it follows that $F_1/\rho d^2 u^2$ and $f/\rho d^2 u^2$ are functions of the Mach number, M, and Reynolds number, R_e, the dimensionless spin, ν, and also of λ, $\bar{\lambda}$, μ, and $\bar{\mu}$, where a bar over a symbol denotes the complex conjugate. It is observed in passing that the notation employed here differs slightly from that of Refs. 1, 8, and 10 in which u_1 replaces u in Eqs. (1) and (2) and in the ratios $F_1/\rho d^2 u^2$ and $f/\rho d^2 u^2$. This in no way affects the analysis to follow and leads to the definition of aerodynamic coefficients consistent with current usage.

At this point, an approximation is made. Since usually $\vec{u}$ and $\vec{\omega}$ are almost parallel to x_1 (the nose of the projectile nearly points along the direction of motion), it follows that

$$|\lambda| = |\bar{\lambda}| << 1, \quad |\mu| = |\bar{\mu}| << 1$$

and it is reasonable to expand the aerodynamic forces in Taylor's series expansions in λ, $\bar{\lambda}$, μ, and $\bar{\mu}$ containing only constant and first order terms, i.e.,

$$\frac{F_1}{\rho d^2 u^2} = a_1 + b_1 \lambda + b_2 \mu + \bar{b}_1 \bar{\lambda} + \bar{b}_2 \bar{\mu} \tag{5}$$

$$\frac{f}{\rho d^2 u^2} = a_2 + c_1 \lambda + c_2 \mu + d_1 \bar{\lambda} + d_2 \bar{\mu} \tag{6}$$

Since F_1 is real, the right side of the first equation is real. The coefficients a_1, b_1, etc., are functions of M, R_e, and ν, and for the present are assumed to be independent of λ, $\bar{\lambda}$, μ, and $\bar{\mu}$. A different interpretation will be given later and this requirement will be relaxed, somewhat.

Forces F_1 and f result from the cross velocity, λ, and the cross angular velocity, μ, and any changes in λ and μ will cause changes in F_1 and f. Rotational symmetry will be exploited by rotating λ and μ, without change in magnitude, through the angle of rotational symmetry, θ. This will not change F_1, but it will induce a similar rotation of f, and it is equivalent to rotating the projectile through an angle, $-\theta$.

If ζ is a vector in the complex x_2, x_3 plane, it can be written as

$$\zeta = r\, e^{i\gamma} \tag{7}$$

If ζ is subjected to a positive rotation through the angle θ about the x_1 axis, it is transformed into

$$\zeta' = r\, e^{i(\gamma+\theta)}$$

or

$$\zeta' = \zeta\, e^{i\theta} \tag{8}$$

Hence, rotation of λ and μ through θ results in the following transformations:

$$\lambda' = \lambda e^{i\theta} \tag{9}$$

$$\mu' = \mu e^{i\theta} \tag{10}$$

$$f' = f e^{i\theta} \tag{11}$$

and

$$F_1' = F_1 \tag{12}$$

where a prime denotes parameters associated with the new positions of λ and μ. The truth of the equation for f' follows from rotational symmetry. The orientation of f' with respect to λ' and μ' is the same as that of f with respect to λ and μ. It follows that Eqs. (5) and (6) must be satisfied by F_1', f', λ', and μ', and so

$$\frac{F_1'}{\rho d^2 u^2} = a_1 + b_1 \lambda' + b_2 \mu' + \bar{b}_1 \bar{\lambda}' + \bar{b}_2 \bar{\mu}' \tag{13}$$

$$\frac{f'}{\rho d^2 u^2} = a_2 + c_1 \lambda' + c_2 \mu' + d_1 \bar{\lambda}' + d_2 \bar{\mu}' \tag{14}$$

Substitution of Eqs. (9) through (12) for λ', μ', f', and F_1 shows that

$$\frac{F_1}{\rho d^2 u^2} = a_1 + b_1 \lambda e^{i\theta} + b_2 \mu e^{i\theta} + \bar{b}_1 \bar{\lambda} e^{-i\theta} + \bar{b}_2 \bar{\mu} e^{-i\theta}$$

$$\frac{f e^{i\theta}}{\rho d^2 u^2} = a_2 + c_1 \lambda e^{i\theta} + c_2 \mu e^{i\theta} + d_1 \bar{\lambda} e^{-i\theta} + d_2 \bar{\mu} e^{-i\theta}$$

and comparison of these relations with Eqs. (5) and (6) reveals that

$$b_1\left(1 - e^{i\theta}\right) = 0$$

$$b_2\left(1 - e^{i\theta}\right) = 0$$

$$a_2\left(1 - e^{-i\theta}\right) = 0$$

$$d_1\left(1 - e^{-2i\theta}\right) = 0$$

$$d_2\left(1 - e^{-2i\theta}\right) = 0$$

Since, by definition, $0 < \theta < \pi$, it follows that

$$b_1 = b_2 = a_2 = d_1 = d_2 = 0$$

and

$$\frac{F_1}{\rho d^2 u^2} = a_1 \tag{15}$$

$$\frac{f}{\rho d^2 u^2} = c_1 \lambda + c_2 \mu \tag{16}$$

Completely analogous arguments apply to the aerodynamic moments and hence

$$\frac{G_1}{\rho d^3 u^2} = e_1 \tag{17}$$

$$\frac{g}{\rho d^3 u^2} = c_3 \lambda + c_4 \mu \tag{18}$$

The expressions above were derived for a system of coordinates fixed in the projectile. Actually, they are valid if the x_2 and x_3 axes are allowed to rotate with respect to the projectile, or vice versa. To prove this statement, suppose that an x_1^*, x_2^*, x_3^* system is defined such that x_1^* coincides with x_1 but the angle between x_2^* and x_2 is ϕ, where ϕ may be a function of time (Fig. 1). In the x_2, x_3 system, the vector ζ may be written

$$\zeta = r\, e^{i\gamma}$$

whereas in the x_2^*, x_3^* system, its representation is

$$\zeta^* = r\, e^{i(\gamma+\phi)}$$

so the transformation equation between the two systems is

$$\zeta^* = \zeta\, e^{i\phi}$$

Then

$$f^* = f e^{i\phi}$$

$$g^* = g e^{i\phi}$$

$$\lambda^* = \lambda e^{i\phi}$$

$$\mu^* = \mu e^{i\phi}$$

Obviously F_1 and G_1 are independent of coordinate systems, and multiplying Eqs. (16) and (18) through by $e^{i\phi}$ yields

$$\frac{f^*}{\rho d^2 u^2} = c_1 \lambda^* + c_2 \mu^*$$

$$\frac{g^*}{\rho d^3 u^2} = c_3 \lambda^* + c_4 \mu^*$$

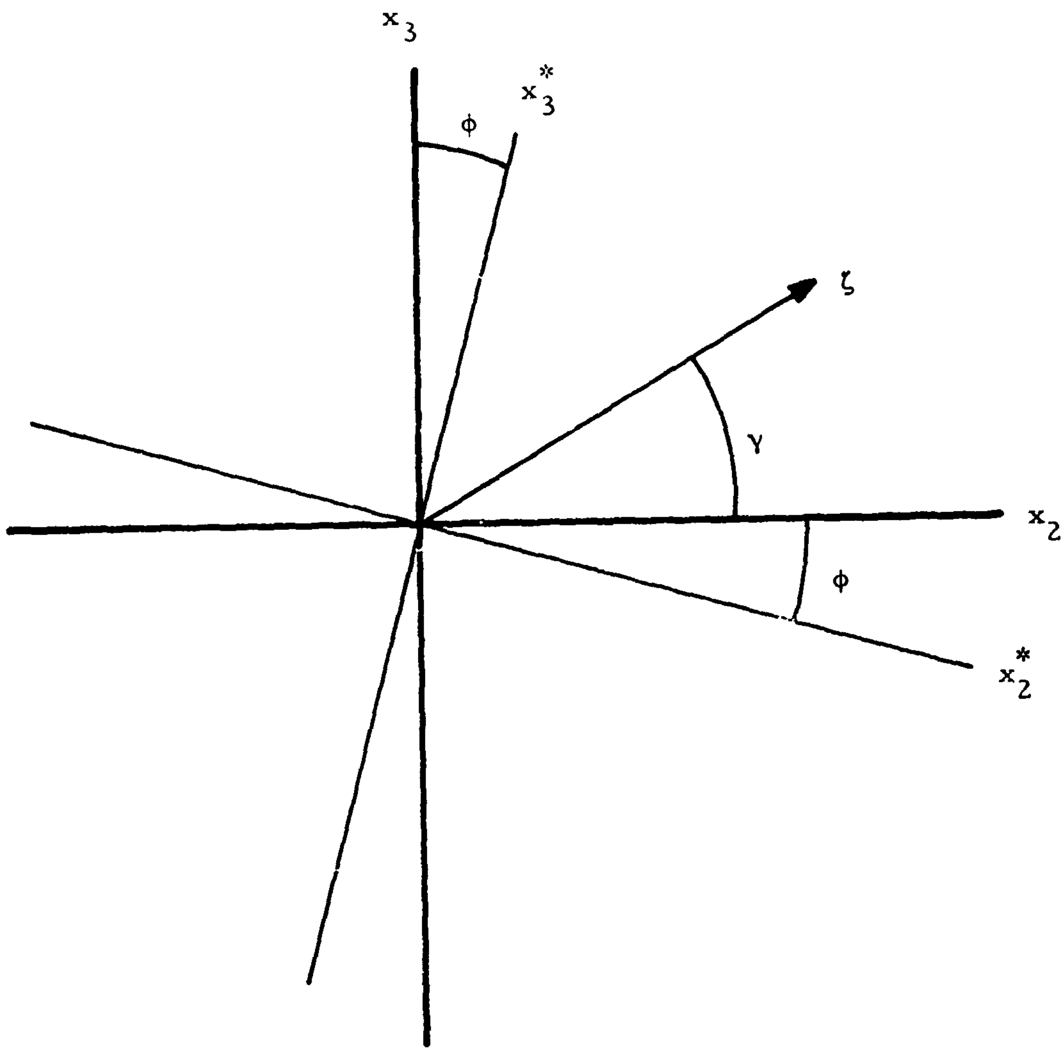


Figure 1

The Relation Between the x_1, x_2, x_3 System and the x_1^*, x_2^*, x_3^* System; $x_1 = x_1^*$ Is out of the Page

Thus, the force and moment equations are valid in the x_1^*, x_2^*, x_3^* system. This is an important simplification.

4. Mirror Symmetry

One might suppose that if the image of a projectile as seen in a mirror were real, its motion would be governed by the same laws of physics as those which determine the motion of the projectile itself. If, in addition, the projectile possesses mirror symmetry, as will be defined, it should be possible to infer the equations of motion of the "image projectile" in terms of those of the projectile itself. By this process, additional information can be gained concerning the aerodynamic forces and moments (Refs. 1 and 18).

The definition of mirror symmetry is as follows (Ref. 1). A projectile possesses mirror symmetry if there is a plane, which contains the longitudinal axis of the projectile, such that, if each point of the projectile is moved to the point on the opposite side of, and equidistant from, the plane, then the projectile exactly covers itself.

It is assumed that the projectile in question possesses a plane of mirror symmetry, in which case the x_1 axis must be in that plane, and it is also convenient to define the x_2 axis as being in that plane.

Consider the following "thought experiment" (Fig. 2) in which a projectile is fired past a mirror. It is arranged so that the projectile plane of mirror symmetry is parallel to the mirror at a given instant of time when observations of the projectile and its image are made. One can choose to imagine the surface of the mirror and the plane of mirror symmetry are coincident but this is not actually necessary. It will be supposed that $\vec{u}$ and $\vec{\omega}$ represent the motion of the projectile whereas $\vec{u}'$ and $\vec{\omega}'$ represent that of the image. Similarly, $\vec{F}$ and $\vec{G}$ represent the aerodynamic force and moment, respectively, acting on the projectile, whereas $\vec{F}'$ and $\vec{G}'$ are associated with the image.

Since x_1 and x_2 lie in the plane of mirror symmetry and x_3 is perpendicular to it, it follows that x_1' and x_2' are parallel to x_1 and x_2,

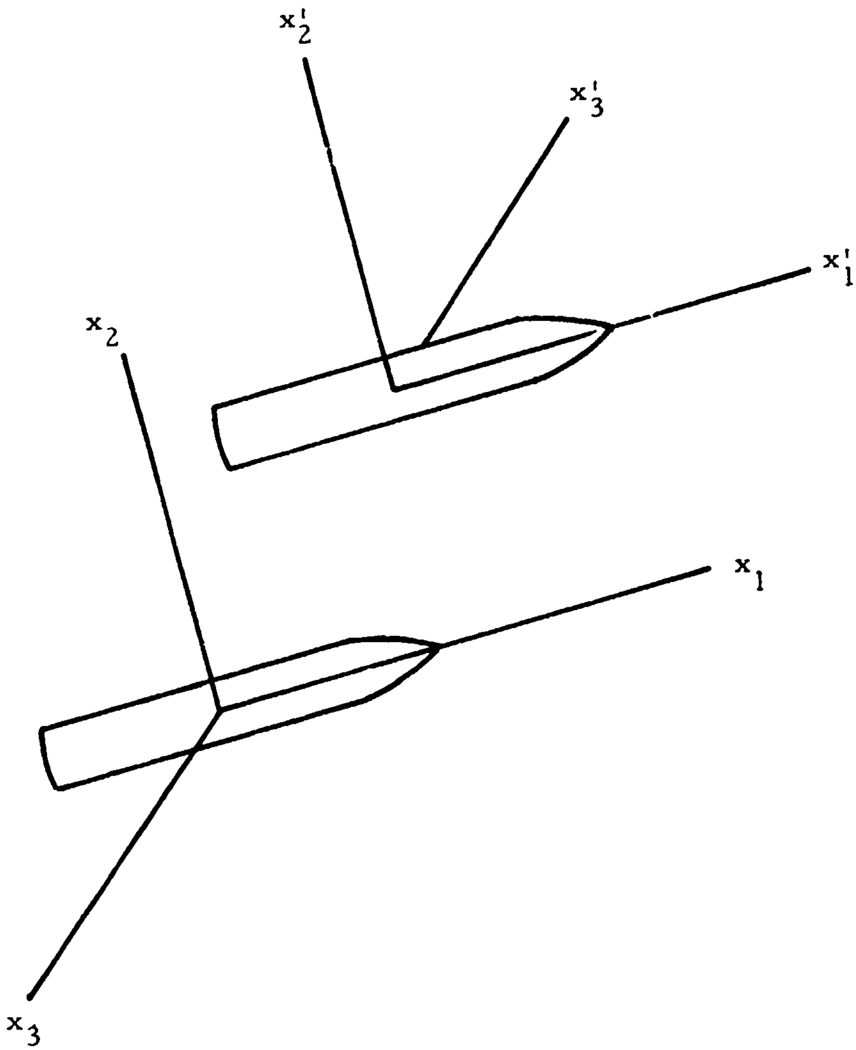


Figure 2
Mirror Symmetry

respectively, and that x_3' is measured in the direction opposite to x_3, where x_1', x_2', x_3' is the coordinate system associated with the image.

Expressions for the components of $\vec{F}'$, $\vec{G}'$, $\vec{u}'$, and $\vec{\omega}'$ in the x_1, x_2, x_3 coordinate system are required in terms of the components of $\vec{F}$, $\vec{G}$, $\vec{u}$, and $\vec{\omega}$. The linear motions of the projectile and its image are the same in the 1- and 2-directions, and so are the forces which cause them; in the 3-direction, the motions and forces are opposite. Hence

$$u_1' = u_1 \qquad F_1' = F_1$$

$$u_2' = u_2 \qquad F_2' = F_2$$

$$u_3' = -u_3 \qquad F_3' = -F_3$$

On the other hand, the angular motions and moments are opposite in the 1- and 2-directions and are the same in the 3-direction, so

$$\omega_1' = -\omega_1 \qquad G_1' = -G_1$$

$$\omega_2' = -\omega_2 \qquad G_2' = -G_2$$

$$\omega_3' = \omega_3 \qquad G_3' = G_3$$

It follows that

$$F_1' = F_1 \tag{19}$$

$$f' = F_2' + iF_3' = F_2 - iF_3 = \bar{f} \tag{20}$$

$$G_1' = -G_1 \tag{21}$$

$$g' = G_2' + i\,G_2' = -G_2 + iG_3 = -\bar{g} \tag{22}$$

$$\lambda' = \frac{u'_2 + i u'_3}{u} = \frac{u_2 - i u_3}{u} = \bar{\lambda} \tag{23}$$

$$\mu' = \frac{(\omega'_2 + i \omega'_3)d}{u} = \frac{(-\omega_2 + i \omega_3)d}{u} = -\bar{\mu} \tag{24}$$

$$\nu' = \frac{\omega'_1 d}{u} = -\frac{\omega_1 d}{u} = -\nu \tag{25}$$

Expressions developed in the preceding section for the aerodynamic forces and moments contain coefficients c_1, c_2, etc., which are functions of M, R_e, and ν. The values of M and R_e are unaffected by reflection through the plane of mirror symmetry, whereas ν changes sign. Accordingly, for the projectile, we write

$$\frac{F_1}{\rho d^2 u^2} = a_1\ (\nu) \tag{26}$$

$$\frac{f}{\rho d^2 u^2} = c_1(\nu)\ \lambda + c_2(\nu)\mu \tag{27}$$

$$\frac{G_1}{\rho d^3 u^2} = e_1(\nu) \tag{28}$$

$$\frac{g}{\rho d^3 u^2} = c_3(\nu)\lambda + c_4(\nu)\mu \tag{29}$$

and for its image,

$$\frac{F'_1}{\rho d^2 u^2} = a_1(\nu') \tag{30}$$

$$\frac{f'}{\rho d^2 u^2} = c_1(\nu')\lambda' + c_2(\nu')\mu' \tag{31}$$

$$\frac{G_1'}{\rho d^3 u^2} = e_1(\nu') \tag{32}$$

$$\frac{g'}{\rho d^3 u^2} = c_3(\nu')\lambda' + c_4(\nu')\mu' \tag{33}$$

Substitution of Eqs. (19) through (25) into Eqs. (30) through (33) yields

$$\frac{F_1}{\rho d^2 u^2} = a_1(-\nu) \tag{34}$$

$$\frac{\overline{f}}{\rho d^2 u^2} = c_1(-\nu)\overline{\lambda} - c_2(-\nu)\overline{\mu} \tag{35}$$

$$\frac{-G_1}{\rho d^3 u^2} = e_1(-\nu) \tag{36}$$

$$\frac{-\overline{g}}{\rho d^3 u^2} = c_3(-\nu)\overline{\lambda} - c_4(-\nu)\overline{\mu} \tag{37}$$

Comparison of this set with Eqs. (26) through (29) leads to the conclusion that

$$a_1(-\nu) = a_1(\nu) \tag{38}$$

$$e_1(-\nu) = -e_1(\nu) \tag{39}$$

$$\overline{c}_1(-\nu) = c_1(\nu) \tag{40}$$

$$\overline{c}_2(-\nu) = -c_2(\nu) \tag{41}$$

$$\overline{c}_3(-\nu) = -c_3(\nu) \tag{42}$$

$$\overline{c}_4(-\nu) = c_4(\nu) \tag{43}$$

Evidently a_1, the real parts of c_1 and c_4, and the imaginary parts of c_2 and c_3 are even functions of ν, whereas e_1, the imaginary parts of c_1 and c_4, and the real parts of c_2 and c_3 are odd functions of ν.

5. The Ballistic K Notation

In the last three subsections, arguments of dimensional analysis and symmetry were used to gain insight into the functional form of the aerodynamic forces and moments acting on a projectile in flight. It was found that the aerodynamic forces and moments can be represented by equations of the form

$$F_1 = \rho d^2 u^2 a_1$$

$$F_2 + iF_3 = \rho d^2 u^2 \left[c_1 \lambda + c_2 \mu\right]$$

$$G_1 = \rho d^3 u^2 e_1$$

$$G_2 + iG_3 = \rho d^3 u^2 \left[c_3 \lambda + c_4 \mu\right]$$

where

$$\lambda = \frac{u_2 + iu_3}{u}$$

$$\mu = \frac{(\omega_2 + i\omega_3)d}{u}$$

$$\nu = \frac{\omega_1 d}{u}$$

and where the dimensionless coefficients a_1, c_1, c_2, etc., are functions of M, R_e, and ν, and satisfy Eqs. (38) through (43) of the last subsection. The only restriction imposed upon the validity of these equations is that the first-order Taylor's series expansions in λ, $\bar{\lambda}$, μ, and $\bar{\mu}$ are adequate, and higher order terms are negligible. For the moment, the truth of this hypothesis is assumed so that the K notation may be introduced.

In the equations above, it is convenient to replace the coefficients a_1, c_1, c_2, etc., by a new set of coefficients in which each term is

divided into its real and imaginary parts, and in which all coefficients are even functions of ν (e.g., if $a(\nu)$ is odd and $\nu b(\nu) = a(\nu)$, then $b(\nu)$ is even). The equations above are written as

$$F_1 = -\rho d^2 u^2 K_{DA} \tag{44}$$

$$F_2 + iF_3 = \rho d^2 u^2 \left[\left\{ -K_N + i\nu K_F \right\} \lambda + \left\{ \nu K_{XF} + iK_S \right\} \mu \right] \tag{45}$$

$$G_1 = -\rho d^3 u^2 \nu K_A \tag{46}$$

$$G_2 + iG_3 = \rho d^3 u^2 \left[\left\{ -\nu K_T - iK_M \right\} \lambda + \left\{ -K_H + i\nu K_{XT} \right\} \mu \right] \tag{47}$$

where the K's are aerodynamic coefficients and are functions of M, R_e, and ν and are even in ν. The K notation is known as the ballistic notation. Signs were chosen so all the K's are positive under normal conditions, but experience has shown that K_T is usually negative.

Separation of real and imaginary parts yields the following equations:

$$\begin{aligned} F_2 = {} & -\rho d^2 u K_N u_2 - \rho d^2 u \nu K_F u_3 \\ & + \rho d^3 u \nu K_{XF} \omega_2 - \rho d^3 u K_S \omega_3 \end{aligned} \tag{48}$$

$$\begin{aligned} F_3 = {} & -\rho d^2 u K_N u_3 + \rho d^2 u \nu K_F u_2 \\ & + \rho d^3 u \nu K_{XF} \omega_3 + \rho d^3 u K_S \omega_2 \end{aligned} \tag{49}$$

$$\begin{aligned} G_2 = {} & -\rho d^3 u \nu K_T u_2 + \rho d^3 u K_M u_3 \\ & -\rho d^4 u K_H \omega_2 - \rho d^4 u \nu K_{XT} \omega_3 \end{aligned} \tag{50}$$

$$\begin{aligned} G_3 = {} & -\rho d^3 u \nu K_T u_3 - \rho d^3 u K_M u_2 \\ & -\rho d^4 u K_H \omega_3 + \rho d^4 u K_{XT} \omega_2 \end{aligned} \tag{51}$$

6. Physical Identification of Forces and Moments

Each term in Eqs. (44) through (47) bears a name and a physical interpretation. Nomenclature is given in Table I. The axial drag, $-\rho d^2u^2K_{DA}$, is obviously the component along the projectile axis of symmetry of the aerodynamic motion-retarding force, whereas the spin decelerating moment, $-\rho d^3u^2\nu K_A$, is the frictional moment which arises from the spin, ω_1, and which tends to damp ω_1.

The normal force, $-\rho d^2u^2K_N\lambda$, is perpendicular to the projectile longitudinal axis (Fig. 3), and is in the direction opposite to the cross velocity, λ. From the definition of λ it follows that

$$\lambda \bar{\lambda} = \sin^2 \delta$$

where δ is the yaw angle, and λu is the component of projectile velocity normal to the projectile longitudinal axis. The plane of yaw is the plane containing $\vec{u}$, λu, and the projectile longitudinal axis, so the normal force is in the plane of yaw.

$\vec{R}$ is the vector sum of the normal force and the axial drag (Fig. 3). Resolution of $\vec{R}$ into components perpendicular and parallel to $\vec{u}$ yields the lift, L, and drag, D, respectively, where

$$L = \rho d^2u^2K_L \sin \delta \tag{52}$$

$$D = -\rho d^2u^2K_D \tag{53}$$

These two forces are equivalent to the normal force and the axial drag, and are more convenient for calculational purposes. It follows from the definitions that

$$K_L = K_N \cos \delta - K_{DA} \tag{54}$$

$$K_D = K_N \sin^2 \delta + K_{DA} \cos \delta \tag{55}$$

Table I

Ballistic Nomenclature

$-\rho d^2 u^2 K_{DA}$	Axial drag
$-\rho d^2 u^2 K_N \lambda$	Normal force
$i\rho d^2 u^2 \nu K_F \lambda$	Magnus force
$i\rho d^2 u^2 K_S \mu$	Cross force due to cross spin
$\rho d^2 u^2 \nu K_{XF} \mu$	Magnus cross force due to cross spin
$-\rho d^3 u^2 \nu K_A$	Spin-decelerating moment
$-\rho d^3 u^2 \nu K_T \lambda$	Magnus moment
$-i\rho d^3 u^2 K_M \lambda$	Overturning moment
$-\rho d^3 u^2 K_H \mu$	Damping moment
$i\rho d^3 u^2 \nu K_{XT} \mu$	Magnus cross torque due to cross spin

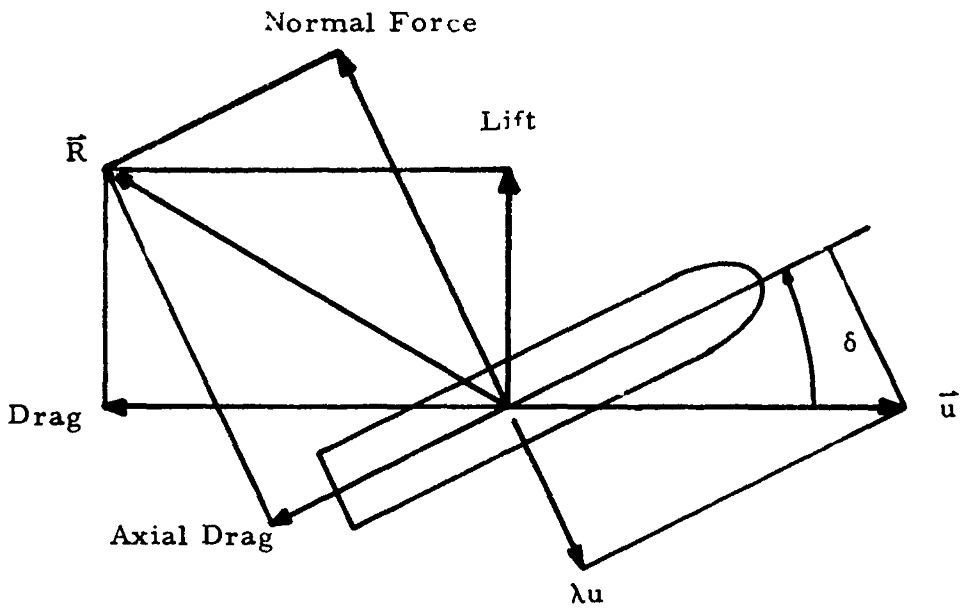


Figure 3
Forces in the Plane of Yaw

The lift and drag are usually used in trajectory computations rather than the normal force and the axial drag.

The overturning moment, $-i\rho d^3 u^2 K_M \lambda$, is the torque which results from the action of the normal force, and its direction is that of $-i\lambda$. It is perpendicular to the plane of yaw and is out of the page in Fig. 3 (not shown). For a spin-stabilized shell, the center of pressure of the normal force is usually ahead (toward the nose) of the center of mass, and the overturning moment usually acts to increase the yaw angle, δ.

The Magnus force, $i\rho d^2 u^2 \nu K_F \lambda$, is perpendicular to the plane of yaw and is into the page in Fig. 3 (not shown). It results from the projectile spin about the longitudinal axis.

The Magnus moment, $-\rho d^3 u^2 \nu K_T \lambda$, is the torque which results from the action of the Magnus force. Its direction is the same as that of the normal force provided K_T is positive.

The cross force due to cross spin, $i\rho d^2 u^2 K_S \mu$, results from the component of cross velocity, μ, as the name implies. Its direction is that of $i\mu$, and is perpendicular to μ and the projectile longitudinal axis. The cross force due to cross spin is physically small and is assumed to have negligible influence on projectile motion. Its presence will be ignored in trajectory calculations.

The damping moment, $-\rho d^3 u^2 K_H \mu$, is the torque which results from the action of the cross force due to cross spin. Its direction is opposite to that of μ, and its action is to damp the cross angular velocity. It has an important influence upon projectile motion.

The Magnus cross force due to cross spin, $\rho d^2 u^2 \nu K_{XF} \mu$, has negligible influence upon projectile motion and is ignored.

Likewise, the Magnus cross torque due to cross spin, $i\rho d^3 u^2 K_{XT} \mu$, which is the moment associated with the Magnus cross force due to cross spin, is negligible.

The reason for retaining the cross force due to cross spin, the Magnus cross force due to cross spin, and the Magnus cross torque due to cross spin in the analysis is for logical consistency. If these terms were deleted, a shift of the projectile center of mass would lead to a logical contradiction. Otherwise, these terms could be excluded from further consideration. The effect of a shift in center-of-mass location is treated in the following subsection.

7. Center-of-Mass Location

Different versions of a particular shell (e.g., HEI, AP, etc.) often have the same external shape, but have different center-of-mass locations. In order that the same aerodynamic data can be used for all versions of a projectile, it is necessary to have appropriate formulae for handling a shift in the center of mass. Such relations are derived in this subsection.

Aerodynamic forces and moments act, respectively, through and about the center of mass of a projectile, and it follows that a shift in the center-of-mass location will produce a different projectile motion. On the other hand, it is clear that aerodynamic pressures and frictional forces are dependent only on the exterior surface of the projectile and its motion relative to the air, and do not depend upon the center-of-mass location. These facts are used to derive the required formulae.

At a given instant of time, it is assumed that $\vec{F}$, $\vec{G}$, $\vec{u}$, and $\vec{\omega}$ are known for a given projectile. If the center of mass is moved forward a distance r along the longitudinal axis to a new position, and if primes denote all parameters defined in a new x_1', x_2', x_3' coordinate system, which is parallel to the old system but with its origin at the new center-of-mass position, then, with respect to the new system

$$\vec{F}' = \vec{F}$$

$$\vec{G}' = \vec{G} - r\vec{x}_1 \times \vec{F}$$

$$= \vec{x}_1 G_1 + \vec{x}_2 (G_2 \quad rF_3) + \vec{x}_3 (G_3 - rF_2)$$

$$\vec{\omega}' = \vec{\omega}$$

$$\vec{u}' = \vec{u} + \vec{\omega} \times r\,\vec{x}_1$$

$$= \vec{x}_1 u_1 + \vec{x}_2(u_2 + r\omega_3) + \vec{x}_3\,(u_3 - r\omega_2)$$

Axial components of $\vec{F}$ and $\vec{G}$ are unchanged, so K_{DA} and K_A are independent of center-of-mass position.

In terms of complex numbers, the transverse components become

$$\lambda' = \frac{u'_2 + iu'_3}{u} = \lambda - ih\mu$$

$$\mu' = \frac{(\omega'_2 + i\omega'_3)d}{u} = \mu$$

$$f' = F'_2 + iF'_3 = f$$

$$g' = G'_2 + iG'_3 = g - ihdf$$

where $h = r/d$ (h is r measured in calibers). It is assumed $u' = u$ since the difference is negligible. The transverse force and moment equations for the new system, Eqs. (16) and (18), become

$$\frac{f'}{\rho d^2 u^2} = c'_1\lambda' + c'_2\mu'$$

$$\frac{g'}{\rho d^3 u^2} = c'_3\lambda' + c'_4\mu'$$

Then

$$c_1'(\lambda - ih\mu) + c_2'\mu = \frac{f}{\rho d^2 u^2} = c_1\lambda + c_2\mu$$

$$c_3'(\lambda - ih\mu) + c_4'\mu = \frac{g - ihdf}{\rho d^3 u^2} = c_3\lambda + c_4\mu - ih(c_1\lambda + c_2\mu)$$

and it follows from a comparison of the coefficients of λ and μ that

$$c_1' = c_1$$

$$c_2' = c_2 + ihc_1$$

$$c_3' = c_3 - ihc_1$$

$$c_4' = c_4 - ih(c_2 - c_3) + h^2 c_1$$

Expressions for the new coefficients in terms of the old may be derived by use of Eqs. (45) and (47). They are:

$K'_{DA} = K_{DA}$	$K'_A = K_A$
$K'_N = K_N$	$K'_M = K_M - hK_N$
$K'_F = K_F$	$K'_T = K_T - hK_F$
$K'_S = K_S - hK_N$	$K'_H = K_H - h(K_S + K_M) + h^2 K_N$
$K'_{XF} = K_{XF} - hK_F$	$K'_{XT} = K_{XT} - h(K_{XF} + K_T) + h^2 K_F$

8. Weaknesses of the Aerodynamic Force and Moment System

The aerodynamic system of forces and moments which is described herein suffers from certain weaknesses. There are at least three areas where one can find fault, namely in arguments based upon projectile symmetry, in the Taylor's series assumption, and the projectile past history. These will be discussed in turn.

The assumption that every projectile possesses perfect rotational and mirror symmetry is obviously incorrect. This is a physical impossibility. One would expect slight deviations from rotational symmetry, and rifling marks on a shell weaken the assumption of mirror symmetry. On the other hand, this is not a very serious problem and, in practice, such deviations from the ideal should produce negligible error on the average. These effects probably give rise to a round-to-round dispersion which may even be desirable. The effect of slight configurational asymmetries is treated in Ref. 13.

The error introduced by the assumption that aerodynamic forces and moments can be represented by a first-order Taylor's series is more important. The Taylor's series assumption implies that the aerodynamic forces and moments are continuous functions with all derivatives, but the characteristics of fluid flow tend to discount this assumption. Instantaneous transitions in flow are observed, such as a breaking away of the flow from a particular location near the nose of the projectile as the yaw angle increases. This suggests a discontinuous nature for aerodynamic forces and moments. Also, the assumption that forces and moments are linear in λ is violated in practice.

There is, fortunately, a way around these difficulties. A close examination of the arguments of symmetry reveals that there is no contradiction if the aerodynamic coefficients (the K's) are allowed to vary with products of the form $\lambda\bar{\lambda}$, $\lambda\bar{\mu}$, $\bar{\lambda}\mu$, and $\mu\bar{\mu}$. These products are invariant under rotation of coordinates about the projectile longitudinal axis and so are the aerodynamic coefficients. Measurements, in fact, show a variation of the aerodynamic coefficients with $\lambda\bar{\lambda} = \sin^2\delta$, but no significant variation with the three other products. It is, accordingly, convenient to drop the Taylor's series assumption, but to retain the equations derived from it as a convenient notation in which the aerodynamic coefficients are allowed to vary with $\sin^2\delta$. Should the need arise, additional correction terms which possess the appropriate symmetry may be added.

The assumption that aerodynamic forces and moments depend only on velocity, $\vec{u}$, and angular velocity, $\vec{\omega}$, is subject to question. A projectile may conceivably reach particular values of $\vec{u}$ and $\vec{\omega}$ by means of a different history of motion. In this instance, the flow pattern of the air around the projectile and the resulting aerodynamic forces and moments may be different.

Aerodynamic forces and moments are dependent upon the pattern of flow of the air mass past a projectile whereas the pattern of flow is in turn a function of the history of motion of the projectile. Presumably, the past history of projectile motion can be reconstructed from knowledge of $\vec{u}$ and $\vec{\omega}$ and all time derivatives of $\vec{u}$ and $\vec{\omega}$ at a given instant. If this is so, it follows that the pattern of flow, and hence the aerodynamic forces and moments, are functions of $\vec{u}$ and $\vec{\omega}$ and all time derivatives of $\vec{u}$ and $\vec{\omega}$. Projectile acceleration and angular acceleration are included in the generalized force system developed in Ref. 10. This system contains the following terms:

$$
\begin{aligned}
f = \rho d^2 u^2 \Big[& \left\{ -K_N + i\nu K_F \right\} \lambda + \left\{ \nu K_{XF} + i K_S \right\} \mu \\
& + \left\{ -K_{NA} + i\nu K_{FA} \right\} \left\{ \lambda' + i \frac{\Omega_1 d}{u} \lambda \right\} \\
& + \left\{ \nu K_{XFA} + i K_{SA} \right\} \left\{ \mu' + i \frac{\Omega_1 d}{u} \mu \right\} \Big]
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
g = \rho d^3 u^2 \Big[& \left\{ -\nu K_T - i K_M \right\} \lambda + \left\{ -K_H + i\nu K_{XT} \right\} \mu \\
& + \left\{ -\nu K_{TA} - i K_{MA} \right\} \left\{ \lambda' + i \frac{\Omega_1 d}{u} \lambda \right\} \\
& + \left\{ -K_{HA} + i\nu K_{XTA} \right\} \left\{ \mu' + i \frac{\Omega_1 d}{u} \mu \right\} \Big]
\end{aligned} \tag{57}
$$

Ω_1 is the component of the angular velocity of the x_1, x_2, x_3 coordinate system in the x_1 direction, and new ballistic coefficients bear the additional subscript A for acceleration. These equations differ from those of Ref. 10 in that u replaces u_1.

Reasonable assumptions as to the size of the acceleration coefficients (Ref. 10) lead to the conclusion that they were all negligible except for K_{MA}. In fact, an approximate solution of the projectile equations of motion reveals that free-flight test range measurements for K_H actually yield $K_H - K_{MA}$ rather than K_H. There is no way to separate K_H and K_{MA} in such measurements.

In conclusion, it should be remarked that these weaknesses in the development do not necessarily imply that ballistics calculations are of insufficient accuracy to be useful. The point is, rather, that ballistics calculations may not be as accurate as one might expect if one is unaware of the weaknesses.

9. The Aerodynamic Notation

As an aid to the free exchange of information between the aerodynamicist and the ballistician, aerodynamic notation has been adapted to projectiles with rotational and mirror symmetry (Refs. 18, 19, and 20). The orthogonal right-handed coordinate system usually employed by the aerodynamicist is a system fixed in the aircraft or missile with the X axis along a principal axis of inertia (the longitudinal axis) and the Y axis parallel to the span of the principal lifting surface (out the right wing). Components of the aerodynamic force and moment in this system are

$$X = \frac{1}{2}\rho V^2 S C_X \qquad L = \frac{1}{2}\rho V^2 S d C_\ell$$

$$Y = \frac{1}{2}\rho V^2 S C_Y \qquad M = \frac{1}{2}\rho V^2 S d C_m$$

$$Z = \frac{1}{2}\rho V^2 S C_Z \qquad N = \frac{1}{2}\rho V^2 S d C_n$$

where X, Y, and Z are components of force, and L, M, and N are components of moment. V is the projectile speed, S is a reference area, and d is a reference length. Only one reference length is used for missiles, contrary to standard usage for aircraft. In the present treatment, the reference length, d, is the projectile diameter and the reference area is

$$S = \frac{\pi}{4} d^2$$

Standard notation for components of velocity and angular velocity in the X, Y, and Z directions are, respectively, u, v, and w, and p, q, and r. The orientation of the velocity $\vec{V}$ in the X, Y, Z coordinate system is defined by the angle of attack, α, and the angle of sideslip, β, where α is measured about the Y axis and β is a rotation about the Z axis. For small angles,

$$\alpha \cong \frac{w}{V} \quad \text{and} \quad \beta \cong \frac{v}{V}$$

Nomenclature for the drag and lift forces is

$$\text{Drag} = \frac{1}{2} \rho V^2 S C_D$$

$$\text{Lift} = \frac{1}{2} \rho V^2 S C_{L\alpha} \alpha$$

and that for the spin deceleration, or damping-in-roll, is

$$L = \frac{1}{2} \rho V^2 S d \frac{pd}{2V} C_{\ell p}$$

The normal coefficients C_Y, C_Z, C_m, and C_n are assumed to be functions of α, β, p, q, r, of time derivatives $\dot{\alpha}$, $\dot{\beta}$, $\dot{q}$, and $\dot{r}$, and are expanded in first-order Taylor's series in terms of these variables. The linear force and moment coefficients which have counterparts in the K notation are defined by the following relations:

$$C_Y = -C_{Y\beta}\beta + C_{Yr}\frac{rd}{2V} - C_{Y\dot\beta}\frac{\dot\beta d}{2V} + C_{Y\dot r}\frac{\dot r d^2}{4V^2}$$

$$+ C_{Yp\alpha}\frac{pd}{2V}\alpha - C_{Ypq}\frac{pd}{2V}\frac{qd}{2V}$$

$$+ C_{Yp\dot\alpha}\frac{pd}{2V}\frac{\dot\alpha d}{2V} - C_{Yp\dot q}\frac{pd}{2V}\frac{\dot q d^2}{4V^2}$$

$$C_Z = -C_{Z\alpha}\alpha + C_{Zq}\frac{qd}{2V} - C_{Z\dot\alpha}\frac{\dot\alpha d}{2V} + C_{Z\dot q}\frac{\dot q d^2}{4V^2}$$

$$+ C_{Zp\beta}\frac{pd}{2V}\beta - C_{Zpr}\frac{pd}{2V}\frac{rd}{2V}$$

$$+ C_{Zp\dot\beta}\frac{pd}{2V}\frac{\dot\beta d}{2V} - C_{Zp\dot r}\frac{pd}{2V}\frac{\dot r d^2}{4V^2}$$

$$C_m = C_{m\alpha}\alpha + C_{mq}\frac{qd}{2V} + C_{m\dot\alpha}\frac{\dot\alpha d}{2V} + C_{m\dot q}\frac{\dot q d^2}{4V^2}$$

$$+ C_{mp\beta}\frac{pd}{2V}\beta + C_{mpr}\frac{pd}{2V}\frac{rd}{2V}$$

$$+ C_{mp\dot\beta}\frac{pd}{2V}\frac{\dot\beta d}{2V} + C_{mp\dot r}\frac{pd}{2V}\frac{\dot r d^2}{4V^2}$$

$$C_n = C_{n\beta}\beta + C_{nr}\frac{rd}{2V} + C_{n\dot\beta}\frac{\dot\beta d}{2V} + C_{n\dot r}\frac{\dot r d^2}{4V^2}$$

$$+ C_{np\alpha}\frac{pd}{2V}\alpha + C_{npq}\frac{pd}{2V}\frac{qd}{2V}$$

$$+ C_{np\dot\alpha}\frac{pd}{2V}\frac{\dot\alpha d}{2V} + C_{np\dot q}\frac{pd}{2V}\frac{\dot q d^2}{4V^2}$$

Terms which have been left out of these expansions are eliminated by the same arguments of symmetry used in the development of the K notation, so it is convenient to ignore them here.

The notation herein differs somewhat from that of the references, and indeed the notation is not in agreement between references. The nomenclature herein has been chosen so that definition of coefficients, particularly signs, are in agreement with those of Arnold Engineering Development Center. This is discussed below.

The aerodynamic coefficients above are sometimes referred to as aerodynamic derivatives; for example, a notation such as

$$C_{Yp\alpha} = \frac{\partial C_Y}{\partial(pd/2V)\,\partial\alpha}$$

is often seen. This notation is strictly correct only if C_Y is indeed a linear function of $(pd/2V)$ and of α.

The Y axis was defined as being parallel to the span of the principal lifting surface for a missile which has such a surface. For a missile with an angle, θ, of rotational symmetry, there may be no such unique surface, and for a shell, θ can be any angle. The Y axis direction is not uniquely defined.

If it is assumed that the projectile has 90-deg rotational symmetry ($\theta = 90^{o}$), this symmetry can be exploited by means of two coordinate systems, e.g., X, Y, Z and X', Y', Z' differing by a rotation of 90 deg about the X axis. It follows that the equations above are valid for both systems and that

$$F_Y' = F_Z \qquad G_Y' = G_Z$$

$$F_Z' = -F_Y \qquad G_Z' = -G_Y$$

$$q' = r \qquad \alpha' = -\beta$$

$$r' = -q \qquad \beta' = \alpha$$

...e relations between forces, moments, angles, and angular rates defined in the two coordinate systems. Utilization of these relations reveals, for example, that

$$C_{Z\alpha} = C_{Y\beta}$$

Table II contains a list of all such relations. Because of the lack of unique Y and Z axis directions, subscripts Y and Z are often replaced by the symbol N (for normal) and m and n are replaced by the symbol M. This notation is also included in Table II, and is called the aeroballistic notation in Ref. 19.

10. Comparison of Ballistic and Aerodynamic Notations

To compare ballistic and aerodynamic nomenclature, it is convenient to use complex number notation. It follows from Table II that

$$\begin{aligned} C_Y + i C_Z = & \\ & - \left[C_{N\alpha} - i \frac{pd}{2V} C_{Np\alpha}\right](\beta + i\alpha) - \left[\frac{pd}{2V} C_{Npq} - i C_{Nq}\right] \frac{(q + ir)d}{2V} \\ & - \left[C_{N\alpha} - i \frac{pd}{2V} C_{Np\dot{\alpha}}\right] \frac{(\dot{\beta} + i\dot{\alpha})d}{2V} - \left[\frac{pd}{2V} C_{Np\dot{q}} - i C_{N\dot{q}}\right] \frac{(\dot{q} + i\dot{r})d^2}{4V^2} \end{aligned} \quad (58)$$

$$\begin{aligned} C_m + i C_n = & \\ & \left[\frac{pd}{2V} C_{Mp\alpha} - i C_{M\alpha}\right](\beta + i\alpha) + \left[C_{Mq} - i \frac{pd}{2V} C_{Mpq}\right] \frac{(q + ir)d}{2V} \\ & + \left[\frac{pd}{2V} C_{Mp\dot{\alpha}} - i C_{M\dot{\alpha}}\right] \frac{(\dot{\beta} + i\dot{\alpha})d}{2V} + \left[C_{M\dot{q}} - i \frac{pd}{2V} C_{Mp\dot{q}}\right] \frac{(\dot{q} + i\dot{r})d^2}{4V^2} \end{aligned} \quad (59)$$

Table II

Equivalent Aerodynamic Coefficients

$C_{Z\alpha}$	=	$C_{Y\beta}$	=	$C_{N\alpha}$
C_{Zq}	=	$-C_{Yr}$	=	C_{Nq}
$C_{Z\dot{\alpha}}$	=	$C_{Y\dot{\beta}}$	=	$C_{N\dot{\alpha}}$
$C_{Z\dot{q}}$	=	$-C_{Y\dot{r}}$	=	$C_{N\dot{q}}$
$C_{Zp\beta}$	=	$-C_{Yp\alpha}$	=	$C_{Np\alpha}$
C_{Zpr}	=	C_{Ypq}	=	C_{Npq}
$C_{Zp\dot{\beta}}$	=	$-C_{Yp\dot{\alpha}}$	=	$C_{Np\dot{\alpha}}$
$C_{Zp\dot{r}}$	=	$C_{Yp\dot{q}}$	=	$C_{Np\dot{q}}$
$C_{m\alpha}$	=	$-C_{n\beta}$	=	$C_{M\alpha}$
C_{mq}	=	C_{nr}	=	C_{Mq}
$C_{m\alpha}$	=	$-C_{n\dot{\beta}}$	=	$C_{M\dot{\alpha}}$
$C_{r\dot{q}}$	=	$C_{n\dot{r}}$	=	$C_{M\dot{q}}$
$C_{mp\beta}$	=	$C_{np\alpha}$	=	$C_{Mp\alpha}$
C_{mpr}	=	$-C_{npq}$	=	C_{Mpq}
$C_{mp\dot{\beta}}$	=	$C_{np\dot{\alpha}}$	=	$C_{Mp\dot{\alpha}}$
$C_{mp\dot{r}}$	=	$-C_{np\dot{q}}$	=	$C_{Mp\dot{q}}$

Also, it is evident that, for small angles,

$$\beta + i\alpha \cong \lambda$$

and

$$\frac{(\dot{\beta} + i\dot{\alpha})d}{2V} \cong \frac{\dot{\lambda}d}{2u} = \frac{\lambda'}{2}$$

where the prime denotes differentiation with respect to arc length, s, where

$$s = \int_0^t \frac{V}{d}dt$$

Also,

$$\frac{(q + ir)d}{2V} = \frac{(\omega_2 + i\omega_3)d}{2u} = \frac{\mu}{2}$$

and

$$\frac{(\dot{q} + i\dot{r})d^2}{4V^2} = \frac{\mu'}{4} + \frac{\mu}{2}\frac{V'}{V} \cong \frac{\mu'}{4}$$

By comparison of the equations above with the corresponding Eqs. (56) and (57), involving the ballistic notation, a one-to-one correspondence can be made between the aerodynamic coefficients and the ballistic coeffieients. This one-to-one correspondence is shown in Table III. Corresponding coefficients are not necessarily equal; if the C notation is used, it should be adopted in toto.

11. Arnold Engineering Development Center (AEDC) Notation

Assume measurements of aerodyramic forces and moments are made in a wind tunnel in which the motion of the projectile model is constrained such that there is no sideslip and there is no rotation about

Table III

Comparison of Ballistic and Aerodynamic Notations

K_D	$\frac{\pi}{8} C_D$
K_L	$\frac{\pi}{8} C_{L\alpha}$
K_N	$\frac{\pi}{8} C_{N\alpha}$
K_F	$\frac{\pi}{16} C_{Np\alpha}$
K_{XF}	$-\frac{\pi}{32} C_{Npq}$
K_S	$\frac{\pi}{16} C_{Nq}$
K_{NA}	$\frac{\pi}{16} C_{N\dot{\alpha}}$
K_{FA}	$\frac{\pi}{32} C_{Np\dot{\alpha}}$
K_{XFA}	$-\frac{\pi}{64} C_{Np\dot{q}}$
K_{SA}	$\frac{\pi}{32} C_{N\dot{q}}$
K_A	$\frac{\pi}{16} C_{\ell p}$
K_M	$\frac{\pi}{8} C_{M\alpha}$
K_T	$-\frac{\pi}{16} C_{Mp\alpha}$
K_{XT}	$-\frac{\pi}{32} C_{Mpq}$
K_H	$-\frac{\pi}{16} C_{Mq}$
K_{MA}	$\frac{\pi}{16} C_{M\dot{\alpha}}$
K_{TA}	$-\frac{\pi}{32} C_{Mp\dot{\alpha}}$
K_{XTA}	$-\frac{\pi}{64} C_{Mp\dot{q}}$
K_{HA}	$-\frac{\pi}{32} C_{M\dot{q}}$

the Z axis, i.e., such that $\beta = 0$, $r = 0$, and $\dot{\alpha} = q$. (The X axis is taken to be at an angle α above the relative wind vector, the Y axis is taken to be horizontal and out the side of the wind tunnel, and the Z axis is directed downward and at an angle α from the vertical.) Under these constraints and with negligible terms excluded, the aerodynamic equations become

$$C_Y = C_{Yp\alpha} \frac{pd}{2V} \alpha$$

$$C_Z = - C_{Z\alpha} \alpha$$

$$C_m = C_{m\alpha} \alpha + (C_{mq} + C_{m\dot{\alpha}}) \frac{qd}{2V}$$

$$C_n = C_{np\alpha} \frac{pd}{2V} \alpha$$

With one exception, these relations define the notation used at AEDC, where $C_{N\alpha}$ is used in place of $C_{Z\alpha}$. It is also evident that C_{mq} and $C_{m\dot{\alpha}}$ cannot be measured separately in a wind tunnel. A similar conclusion has already been mentioned with regard to free-flight test measurements since

$$K_H - K_{MA} = - \frac{\pi}{16} \left(C_{mq} + C_{m\dot{\alpha}}\right)$$

(See Subsection 8.)

It was mentioned that the notation herein was chosen to be in agreement with that of AEDC. The following comments seem appropriate: AEDC reports (Refs. 30 through 38) available to the author contain data for $C_{N\alpha}$, $C_{Yp\alpha}$, $C_{m\alpha}$, $C_{mq} + C_{m\dot{\alpha}}$, and $C_{np\alpha}$. Data for the other coefficients (besides drag and damping-in-roll) is lacking and for them no definite sign convention is available. These missing

coefficients are considered to be negligible, however, and no sign convention is needed. Numerical values of $C_{N\alpha}$ and $C_{m\alpha}$ are positive, whereas $C_{mq} + C_{m\dot{\alpha}}$ is negative. Signs of Magnus coefficients $C_{Yp\alpha}$ and $C_{np\alpha}$ may be in error since it is difficult to determine their sign conventions from AEDC reports. It is noted that AEDC does not strictly adhere to these definitions. One should be very careful to obtain correct information as to definitions of force and moment coefficients from wind tunnel and free-flight test range personnel. Their definitions may differ somewhat from those herein, and it is very easy to get a sign wrong or, say, be off by a factor of two. Some people use pd/V in their definitions rather than pd/2V.

Reference 39 (Advisory Group For Aerospace Research and Development, AGARDograph 121) contains methods of obtaining aerodynamic data by use of wind tunnels. The notation herein agrees with that defined in Ref. 39 except for C_{Nq} which is negligible.

The notation herein differs somewhat from that of Ref. 40 (Ballistics Research Laboratory Memorandum Report No. 2192). The major difference is in the use of pd/V instead of pd/2V for the dimensionless spin.

12. Modeling of Aerodynamic Data for Trajectory Computations

In previous subsections, expressions have been developed for aerodynamic forces and moments in terms of the aerodynamic coefficients, but nothing has been said about the form of these coefficients other than that they are functions of M, R_e, ν, and δ. Attempts have been made to derive expressions for these coefficients from fundamental theory, but in practice, it is necessary to measure them for various combinations of M, R_e, ν, and δ. Such measurements are made in wind tunnels, or by means of free-flight tests (Refs. 30 through 47). Data thus obtained is tabular and a problem arises in the utilization of this data in trajectory computation on a digital computer. Means of multi-dimensional interpolation between data points is required. To ascertain the extent of the problem, it is advantageous to examine the

tabular data and determine as much as possible about the functional form of the aerodynamic coefficients.

12.1 The Functional Form of the Aerodynamic Coefficients - As has been stated, the aerodynamic coefficients are functions of Mach number, M, Reynolds number, R_e, dimensionless spin, ν, and the yaw angle, δ. Fortunately, variations with R_e are slight and usually can be ignored. The expression for R_e is

$$R_e = \frac{\rho u d}{\mu_a} = \frac{M V_s d}{\eta_a}$$

where η_a is the kinetic viscosity and V_s is the speed of sound. Since both V_s and μ_a are functions of altitude, R_e varies with altitude and attempts have been made to model certain aerodynamic data as such (Ref. 48). Unless extreme variations in altitude along a trajectory are anticipated, however, Reynolds number variations may be ignored. This may be regarded as the best policy unless experience proves otherwise.

The effect of dimensionless spin, ν, upon the drag, lift, and overturning moment coefficients, K_D, K_L, and K_M, respectively, is believed to be negligible. That there is a slight effect is demonstrated in Ref. 7. Until proven otherwise, K_D, K_L, and K_M are assumed to be independent of ν. The damping moment coefficient K_H is also usually modeled as such. On the other hand, the Magnus coefficients, K_F and K_T, are both functions of ν and should be modeled as such. The spin deceleration coefficient, K_A, is presumably also a function of ν, but K_A is hard to measure and little data may be available. This poses somewhat of a dilemma since lack of knowledge of K_A in trajectory computation implies a lack of knowledge of ν. Extensive modeling of K_F and K_T in terms of ν is then of questionable value.

All coefficients are functions of M and δ, although adequate data for modeling may be hard to obtain. This is particularly true of K_F and K_T since they are functions of three variables, M, ν, and δ.

In summary, the aerodynamic coefficients may be written in terms of their independent variables as follows:

$$K_D = K_D(M, \delta)$$

$$K_L = K_L(M, \delta)$$

$$K_M = K_M(M, \delta)$$

$$K_H = K_H(M, \delta)$$

$$K_F = K_F(M, \nu, \delta)$$

$$K_T = K_T(M, \nu, \delta)$$

$$K_A = K_A(M, \nu, \delta)$$

All other aerodynamic forces and moments are considered to be negligible, and are excluded from consideration.

12.2 Polynomial Curve Fits - Modeling of aerodynamic data for computer use can be accomplished by fitting low-order polynomials in two or three variables to the data. The drag coefficient can usually be expressed by the relation

$$K_D = K_{D_o}(M)\left[1 + K_{D_{\delta^2}}(M)\sin^2\delta\right]$$

in which $K_{D_o}(M)$ is the zero-yaw drag and the term in the square brackets accounts for non-zero yaw. For small-yaw applications, $\sin\delta$ is sometimes replaced with δ, whereas for high-yaw situations, more terms in $\sin^4\delta$, $\sin^6\delta$, etc. can be included if they are needed. $K_{D_o}(M)$ and $K_{D_{\delta^2}}(M)$ may be fitted in sections in powers of M or 1/M. If M_1, $M_2, \ldots, M_n$ are numbers such that

$$M_1 < M_2 < \ldots < M_n$$

for example,

$$K_{D_o} = a_i + \frac{b_i}{M}$$

provides a good fit to K_{D_o} over the interval

$$M_i < M \leq M_{i+1}$$

if the interval is not too large. On the other hand, an expression valid for high Mach numbers has been found useful. It is

$$K_{D_o} = \frac{(a + bM)^2 - 1}{M^2}$$

This can be obtained by curve fitting the "Q function" (Ref. 49).

$$Q = \sqrt{1 + M^2 K_{D_o}} = a + b\,M$$

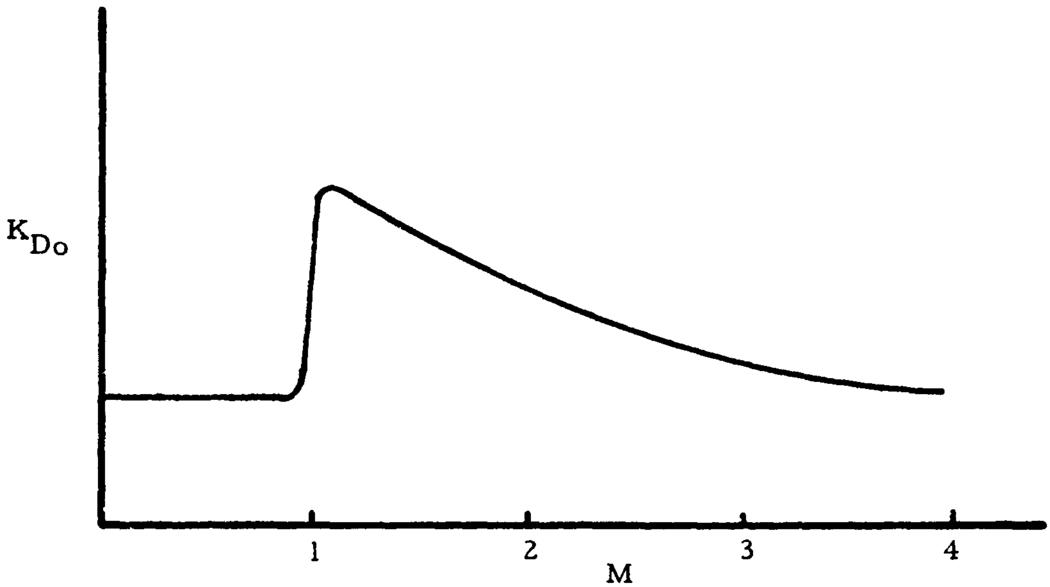
The lift coefficient, K_L, may be handled in a similar fashion. In the interval $M_i < M \leq M_{i+1}$ and $\sin \delta_i < \sin \delta \leq \sin \delta_{i+1}$ one can write, for example

$$\begin{aligned} K_L = {} & (a_0 + a_1 M + \cdots + a_\ell M^\ell) \\ & + (a_0 + b_1 M + \cdots + b_m M^m) \sin^2 \delta \\ & + \cdots \\ & + (c_0 + c_1 M + \cdots + c_n N^n) \sin^{2k} \delta \end{aligned}$$

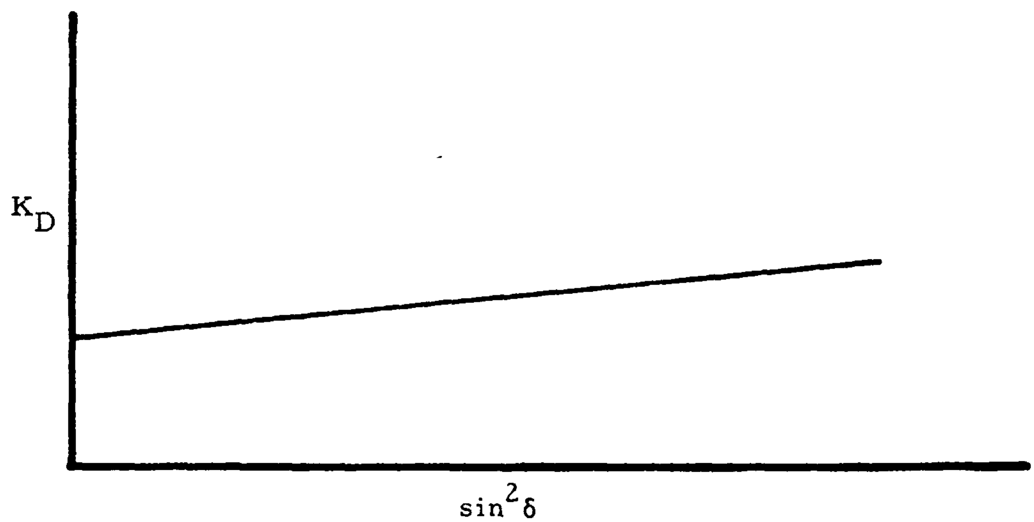
Similar expressions can be written for K_M and K_H.

K_F, K_T, and K_A are functions of three variables: M, ν, and $\sin^2 \delta$. Polynomials may be constructed for them in much the same manner.

..ical plots of K_{D_o} vs M and K_D vs $\sin^2\delta$ with M constant are shown in Fig. 4. Plots of other aerodynamic coefficients may be found in the literature. See, for example, Refs. 30 through 40. Three-dimensional plots of some of the coefficients are shown in Ref. 50.



(a) Zero-Yaw Drag Coefficient



(b) Plot of K_D vs $\sin^2\delta$ for M Constant

Figure 4
The Drag Coefficient

SECTION III

A STUDY OF PROJECTILE ANGULAR MOTION

1. General

The system of aerodynamic forces and moments has been subjected to a comprehensive examination in the previous section and it is convenient at this point to investigate the effect of these forces and moments upon the motion of a spinning shell. There is no hope of obtaining a closed-form analytical solution for this motion, so it is appropriate to solve simplified sets of equations to gain insight into the behavior of projectile motion. The starting place is to solve the equations of motion for the case where there are no aerodynamic forces and moments acting upon the shell. The main interest, here, is in the angular motion, since the translational motion is simple. The next step is to solve the equations of angular motion for the case where the only torque acting is an overturning moment. These two simplified examples aid in understanding the precessional and nutational motions of a projectile. The second case involving the overturning moment is mathematically equivalent to the motion of a spinning top acted upon by gravity, as is treated in many text books on Mechanics (Ref. 51, for example). The next step is to compare the motion of a projectile acted upon by an overturning moment with epicyclic motion, which is sometimes used to approximate projectile motion. The addition of damping to epicyclic motion completes the picture, more or less, and gives rise to an understanding of dynamic stability.

With this background in mind, it is possible to obtain an approximate solution for the angular motion of a spinning shell by use of the complex notation of Section II. The six-degree-of-freedom equations of motion are set up in the complex notation and solved under simplifying assumptions as to the importance of certain terms with regard to size. Slowly varying terms are assumed to be constant, so the solution should be valid only along a short section of the trajectory. The solution thus obtained is epicyclic.

This section is closed with a discussion of windage jump and drift, both of which result from the angular motion of the projectile and forces normal to the trajectory. Windage jump must be considered in situations where the angle between the gun and the aircraft velocity vector is large. Drift is not usually considered in air firings, but it may be important for new rounds under development at long ranges.

The material in this section is entirely tutorial. The equations derived using the complex notation are not suitable for numerical integration, and equations appropriate for that purpose are derived in Section III. A knowledge of this material is essential to the understanding of projectile motion, windage jump, and drift, however.

2. The Equations of Motion

The six-degree-of-freedom equations of motion are derived from Newton's laws, which are

$$m\frac{d\vec{u}}{dt} = \vec{F} + m\vec{g}$$

$$\frac{d\vec{H}}{dt} = \vec{G}$$

where m is the projectile mass, $\vec{g}$ is the acceleration due to gravity, and $\vec{H}$ is the angular momentum. The x_1, x_2, x_3 coordinate system of the projectile is chosen in an incompletely specified manner so that the longitudinal axis of the projectile coincides with the x_1 axis, but the projectile is free to rotate with respect to the x_2, x_3 axes. The angular velocity of the x_1, x_2, x_3 coordinate system is $\vec{\Omega}$ and it follows that

$$\omega_2 = \Omega_2$$

$$\omega_3 = \Omega_3$$

The axial and transverse moments of inertia of the projectile are A and B, respectively, and, with respect to the x_1, x_2, x_3 coordinate system

$$\vec{H} = \vec{x}_1 A\omega_1 + \vec{x}_2 B\omega_2 + \vec{x}_3 B\omega_3 \tag{60}$$

where $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are unit vectors in the indicated directions. The time rate of change of a vector $\vec{C}$ in a rotating coordinate system with angular velocity $\vec{\Omega}$ is given by

$$\dot{\vec{C}} = \underset{\cdot}{\vec{C}} + \vec{\Omega} \times \vec{C} \tag{61}$$

where a dot above the arrow indicates a time derivative measured in inertial space and a dot below the arrow represents a time derivative observed in the rotating coordinate system (Ref. 51). It follows that

$$m(\dot{u}_1 + \omega_2 u_3 - \omega_3 u_2) = F_1 + mg_1 \tag{62}$$

$$m(\dot{u}_2 + \omega_3 u_1 - \Omega_1 u_3) = F_2 + mg_2 \tag{63}$$

$$m(\dot{u}_3 + \Omega_1 u_2 - \omega_2 u_1) = F_3 + mg_3 \tag{64}$$

$$A\dot{\omega}_1 = G_1 \tag{65}$$

$$B\dot{\omega}_2 + \omega_3 A\omega_1 - \Omega_1 B\omega_3 = G_2 \tag{66}$$

$$B\dot{\omega}_3 + \Omega_1 B\omega_2 - \omega_2 A\omega_1 = G_3 \tag{67}$$

Use has been made of $\Omega_2 = \omega_2$ and $\Omega_3 = \omega_3$. The set of angular motion equations will be solved for the case where $\vec{G} = 0$, and for the case where $G_1 = G_2 = 0$ and $G_3 = M \sin \theta$. This will be followed by a discussion of epicyclic motion, damped epicyclic motion, and dynamic

stability. These solutions should give the reader sufficient insight into the angular motion of a projectile to understand later developments.

3. Torque-Free Motion

It is of interest in passing to ascertain the motion of a projectile when the aerodynamic torque is zero. In this instance, the angular momentum vector $\vec{H}$ is constant in magnitude and direction, and it will be convenient to choose a coordinate system such that the x axis lies along $\vec{H}$ (Fig. 5). In such a system, the motion of the projectile is particularly simple.

In Fig. 5, θ is the angle between the projectile x_1 axis and x axis, and ϕ is the angle between the y axis and the plane containing the x_1 and x_2 axes. The angular velocity of the projectile is

$$\vec{\omega} = \vec{x}_1 \dot{\psi} + \vec{1}_x \dot{\phi} + \vec{x}_3 \dot{\theta}$$

where $\dot{\psi}$ is the rate of change of the orientation angle of the projectile measured about the x_1 axis, and $\vec{1}_x$ is a unit vector in the x direction. But

$$\vec{1}_x = \vec{x}_1 \cos\theta - \vec{x}_2 \sin\theta$$

and the components of $\vec{\omega}$ in the x_1, x_2, x_3 system are

$$\omega_1 = \dot{\psi} + \dot{\phi}\cos\theta \tag{68}$$

$$\omega_2 = -\dot{\phi}\sin\theta \tag{69}$$

$$\omega_3 = \dot{\theta} \tag{70}$$

The components of angular momentum are

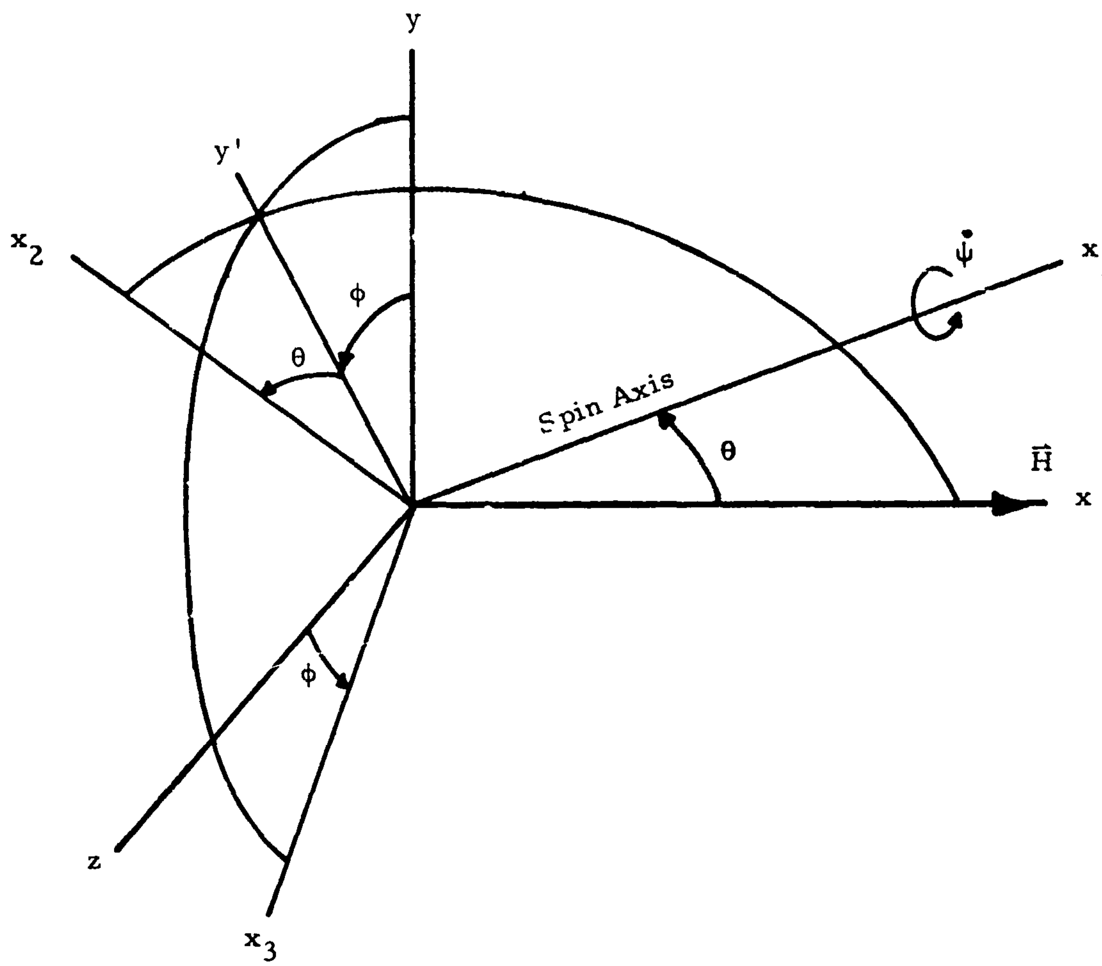


Figure 5
Coordinates for Studying Angular Motion

$$H_1 = A\omega_1 \tag{71}$$

$$H_2 = B\omega_2 \tag{72}$$

$$H_3 = B\omega_3 \tag{73}$$

Since x_3 is perpendicular to $\vec{H}$, it follows that

$$H_3 = B\omega_3 = B\dot{\theta} = 0 \tag{74}$$

so

$$\omega_3 = \dot{\theta} = 0$$

and θ is constant. Components of

$$\vec{H} = \vec{x}_1 A\omega_1 + \vec{x}_2 B\omega_2$$

resolved along the x and y' directions are

$$A\omega_1 \cos\theta - B\omega_2 \sin\theta = H \tag{75}$$

$$A\omega_1 \sin\theta + B\omega_2 \cos\theta = 0 \tag{76}$$

Since θ is constant, ω_1 and ω_2 are also, because this set of equations can be solved simultaneously for ω_1 and ω_2 in terms of θ and H. Since ω_2 is constant, Eq. (69) shows that

$$\dot{\phi} = -\frac{\omega_2}{\sin\theta} \tag{77}$$

is constant, and it follows from Eq. (68) that $\dot{\psi}$ is also, since ω_1 is constant.

The analysis is almost complete for a projectile which is acted upon by no aerodynamic torques. It precesses about its angular momentum vector with constant rate, $\dot{\phi}$. An expression for $\dot{\phi}$ in terms of H is required, however. Since $\dot{\omega}_3 = 0$, it follows from Eq. (67) that

$$\Omega_1 B\omega_2 - \omega_2 A\omega_1 = \omega_2(B\Omega_1 - A\omega_1) = 0$$

and since, in general, $\omega_2 \neq 0$

$$\Omega_1 = \frac{A}{B}\omega_1 \tag{78}$$

But Ω_1 is the angular velocity of the x_1, x_2, x_3 coordinate system about x_1 and

$$\omega_1 = \dot{\psi} + \Omega_1$$

so, from Eq. (68)

$$\Omega_1 = \dot{\phi}\cos\theta \tag{79}$$

From Eqs. (75) and (76)

$$A\omega_1 = H\cos\theta \tag{80}$$

From Eqs. (78), (79), and (80), it follows that

$$\dot{\phi} = \frac{H}{B} \tag{81}$$

It also follows that

$$\omega_1 = \frac{H}{A} \cos \theta \tag{82}$$

$$\omega_2 = -\frac{H}{B} \sin \theta \tag{83}$$

so ω_1 and ω_2 are given as functions of θ.

4. Motion Under the Action of an Overturning Moment

A second example of interest is the motion of a projectile under the influence of an overturning moment. Figure 5 and the notation of the last subsection can be used, but $\vec{H}$ is no longer along the x axis as shown. The overturning moment is always perpendicular to the plane containing x_1 and x and is taken to be

$$G_3 = M \sin \theta \tag{84}$$

where M is a positive constant.

This problem is mathematically equivalent to the motion of a spinning top under the influence of gravity, in which case $M = mg\ell$, where ℓ is the distance from the top center of mass to the pivot (Ref. 51).

By Eq. (65), with $G_1 = 0$

$$H_1 = A\omega_1 \tag{85}$$

is constant. Also, since G_3 is perpendicular to x, the component of $\vec{H}$ in the x direction,

$$H_x = A\omega_1 \cos \theta - B\omega_2 \sin \theta \tag{86}$$

is constant. These last two equations yield

$$\omega_2 = \frac{H_1 \cos \theta - H_x}{B \sin \theta} \tag{87}$$

and from Eq. (69)

$$\dot{\phi} = -\frac{H_1 \cos\theta - H_x}{B \sin^2\theta} \tag{88}$$

From Eqs. (68), (85), and (88)

$$\dot{\psi} = \frac{H_1}{A} + \frac{H_1 \cos\theta - H_x}{B \sin^2\theta} \cos\theta \tag{89}$$

These expressions for $\dot{\phi}$ and $\dot{\psi}$ are dependent upon θ alone. An expression for $\dot{\theta}$ in terms of θ can be derived from consideration of the energy. The rate of doing work is $\vec{G} \cdot \vec{\omega}$, and from Eq. (61),

$$\dot{\vec{H}} = \vec{\dot{H}} + \vec{\Omega} \times \vec{H} = \vec{G}$$

It follows that

$$\dot{\vec{H}} \cdot \vec{\omega} = (\vec{\dot{H}} + \vec{\Omega} \times \vec{H}) \cdot \vec{\omega} = \vec{\dot{H}} \cdot \vec{\omega} = \vec{G} \cdot \vec{\omega}$$

since

$$\vec{\omega} \cdot (\vec{\Omega} \times \vec{H}) = \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \Omega_1 & \omega_2 & \omega_3 \\ A\omega_1 & B\omega_2 & B\omega_3 \end{vmatrix} = 0$$

But, according to Eq. (60)

$$\dot{\vec{H}} \cdot \vec{\omega} = A\dot{\omega}_1\omega_1 + B\dot{\omega}_2\omega_2 + B\dot{\omega}_3\omega_3$$

$$= \frac{d}{dt} \frac{1}{2} \left(A\omega_1^2 + B\omega_2^2 + B\omega_3^2\right)$$

and from Eqs. (70) and (84), and since $G_1 = G_2 = 0$

$$\vec{G} \cdot \vec{\omega} = G_3\omega_3 = \dot{\theta} M \sin\theta = -\frac{d}{dt} M \cos\theta$$

Then

$$\frac{d}{dt} \frac{1}{2} (A\omega_1^2 + B\omega_2^2 + B\omega_3^2) = -\frac{d}{dt} M \cos\theta$$

and it follows that

$$\frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + B\omega_3^2\right) = -M\cos\theta + E' \tag{90}$$

where E' is a constant. Since ω_1 is constant, this expression may be written as

$$\frac{1}{2}B\left(\omega_2^2 + \omega_3^2\right) + M\cos\theta = E' - \frac{1}{2}A\omega_1^2 = E \tag{91}$$

where E is a constant. E', of course, is the total rotational energy of the projectile, and E is the energy associated with the transverse motion. Solution of this expression for ω_3^2 yields

$$\omega_3^2 = \frac{2E}{B} - \frac{2M}{B}\cos\theta - \omega_2^2$$

and from Eqs. (70) and (87),

$$\dot{\theta}^2 = \frac{2E}{B} - \frac{2M}{B} \cos\theta - \left(\frac{H_1 \cos\theta - H_x}{B \sin\theta}\right)^2 \tag{92}$$

This is the desired expression for $\dot{\theta}$ in terms of θ. With the substitution

$$w = \cos\theta \tag{93}$$

and with rearrangement, it can be put in the form

$$\dot{w}^2 = (1 - w^2)(\alpha - \beta w) - (aw - b)^2 = F(w) \tag{94}$$

where α, β, a, and b are constants given by

$$\alpha = \frac{2E}{B} \tag{95}$$

$$\beta = \frac{2M}{B} \tag{96}$$

$$a = \frac{H_1}{B} \tag{97}$$

$$b = \frac{H_x}{B} \tag{98}$$

It follows that

$$t = \int_{w_o}^{w} \frac{dw}{\sqrt{F(w)}} \tag{99}$$

A solution for t may be found in terms of elliptic functions according to Ref. 51, but such a solution would serve no particular purpose here. It is sufficient to make certain observations concerning the variation of θ with time. $F(w)$ is a cubic and may be plotted as shown in Fig. 6. For large $|w|$, the dominant term in $F(w)$ is βw^3, and since β is positive, $F(w)$ is positive for large positive w and negative for large negative w. For $w = \pm 1$, $F(w)$ equals $-(\pm a - b)^2$ and is negative (unless $b = a$ or $b = -a$). It follows that the plot of $F(w)$ vs w must have the characteristics shown in Fig. 6, with two roots in the region $-1 < w < 1$, and a third root in the region $w > 1$. The physical motion of the projectile can occur only if $F(w)$ is positive, and this occurs between w_1 and w_2. Otherwise, $\dot{\theta}^2$ would be negative and $\dot{\theta}$ imaginary, or else $\cos\theta$ would be greater than zero, which is impossible. Thus

$$w_1 = \cos\theta_1 \le \cos\theta \le \cos\theta_2 = w_2$$

or

$$\theta_2 \le \theta \le \theta_1$$

The angles θ_1 and θ_2 are "turning angles" at which

$$\dot{\theta} = \pm\sqrt{F(\cos\theta)}$$

changes sign. When θ reaches θ_1, $\dot{\theta}$ changes sign and approaches θ_2; θ changes sign again at θ_2 and θ approaches θ_1. This motion is cyclic, and typical examples are sketched in Fig. 7 for different values of parameters α, β, a, and b. These sketches represent $\theta(t)$ vs $\phi(t)$ in polar coordinates as t varies. The projectile is observed to move with a relatively slow precessional motion about the x axis. On top of this precessional motion, a faster "nutational" motion is observed as θ moves back and forth between θ_1 and θ_2.

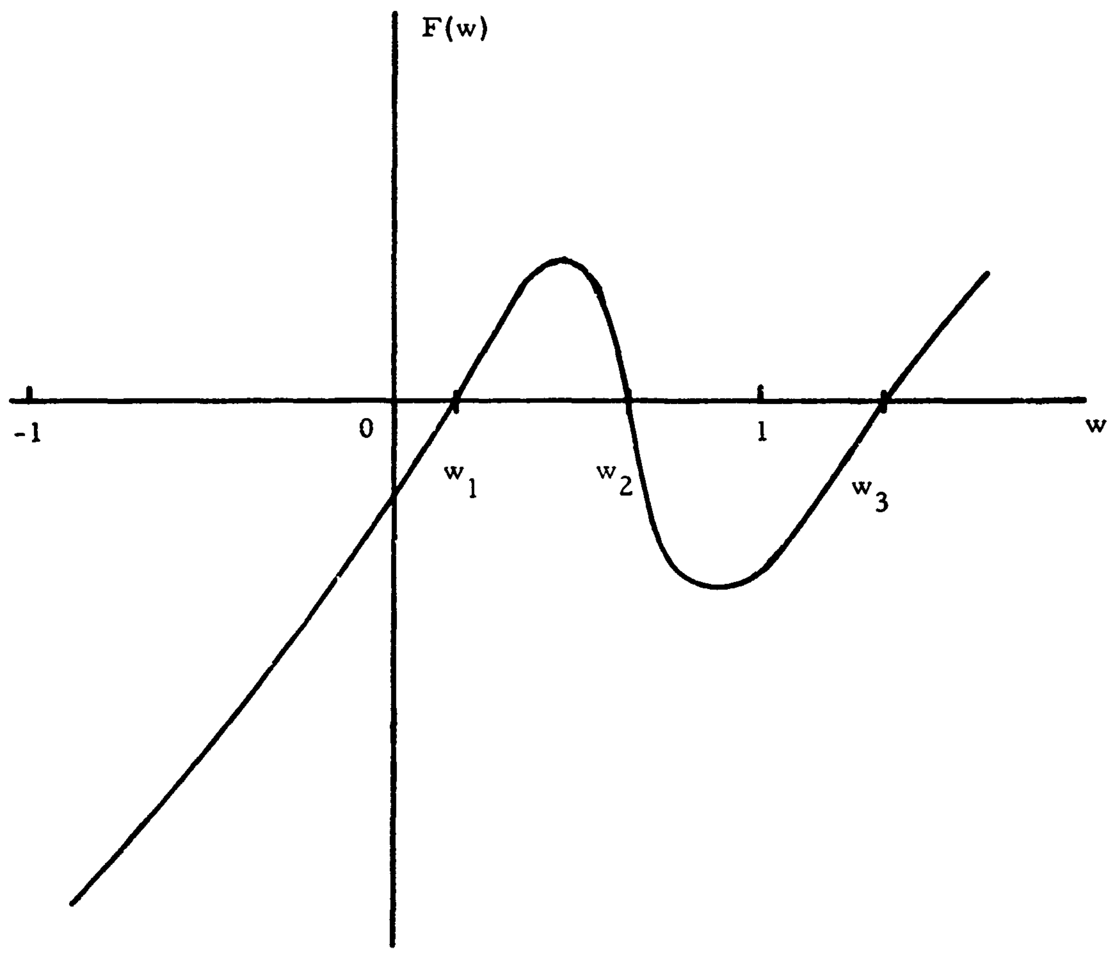
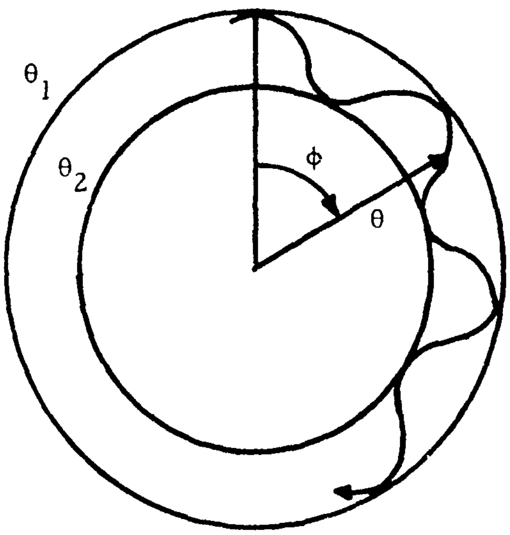
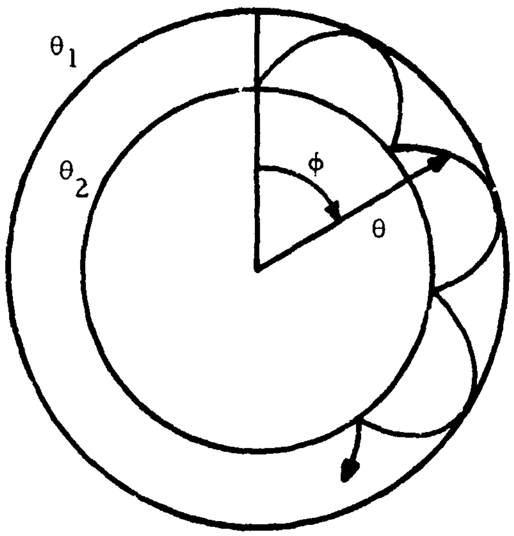


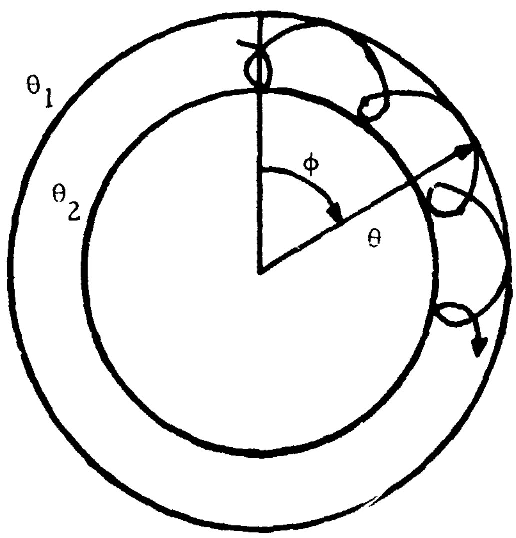
Figure 6
Plot Showing the Characteristics of F(w) vs w



(a) $b - a \cos\theta > 0$
for $\theta_1 \geq \theta \geq \theta_2$



(b) $b - a \cos\theta = 0$ at θ_2 where the motion stops instantaneously



(c) $b - a \cos\theta$ changes sign between θ_1 and θ_2

Figure 7

Typical Examples of Precessional and Nutational Motion

The motion shown in the three examples of Fig. 7 is explained as follows. Equation (88) may be written

$$\dot{\phi} = \frac{b - aw}{1 - w^2}$$

where a and b are determined by initial conditions. In Fig. 7(a), $\dot{\phi}$ is positive, ϕ continuously increases, and the curve is tangent to both the inner and outer circles. In this case, $b - aw > 0$ for all w between w_1 and w_2 (or for θ between θ_1 and θ_2). In Fig. 7(c), $\dot{\phi}$ changes sign between θ_1 and θ_2. Clockwise motion is exhibited at θ_1 but counter-clockwise motion occurs at θ_2. In this case, $b - aw = 0$ has a root between w_1 and w_2; i.e., $w_1 < b/a < w_2$. In Fig. 7(b), $b - aw = 0$ at θ_2, in which case both $\dot{\phi}$ and $\dot{\theta}$ are zero at the same time, the motion stops instantaneously, and a cusp touches the inner circle. This is not an exceptional case, as one might think, since the values of a and b are determined by initial conditions. At the instant a shell is fired from a stationary gun, $\dot{\phi} = \dot{\theta} = 0$ and the expected motion is that of Fig. 7(b).

5. Epicyclic Motion and Dynamic Stability

A type of motion which approximates that described in the last subsection, and which is sometimes used to approximate the angular motion of a projectile, is epicyclic motion. If a wheel is attached to the rim of another wheel, as shown in Fig. 8, a point P on the rim of the first wheel executes epicylic motion. From Fig. 8 it is seen that the coordinates of P are

$$\theta_y = A_p \cos \omega_p t + A_n \cos \omega_n t = |\theta| \cos \phi$$

$$\theta_z = A_p \sin \omega_p t + A_n \sin \omega_n t = |\theta| \sin \phi$$

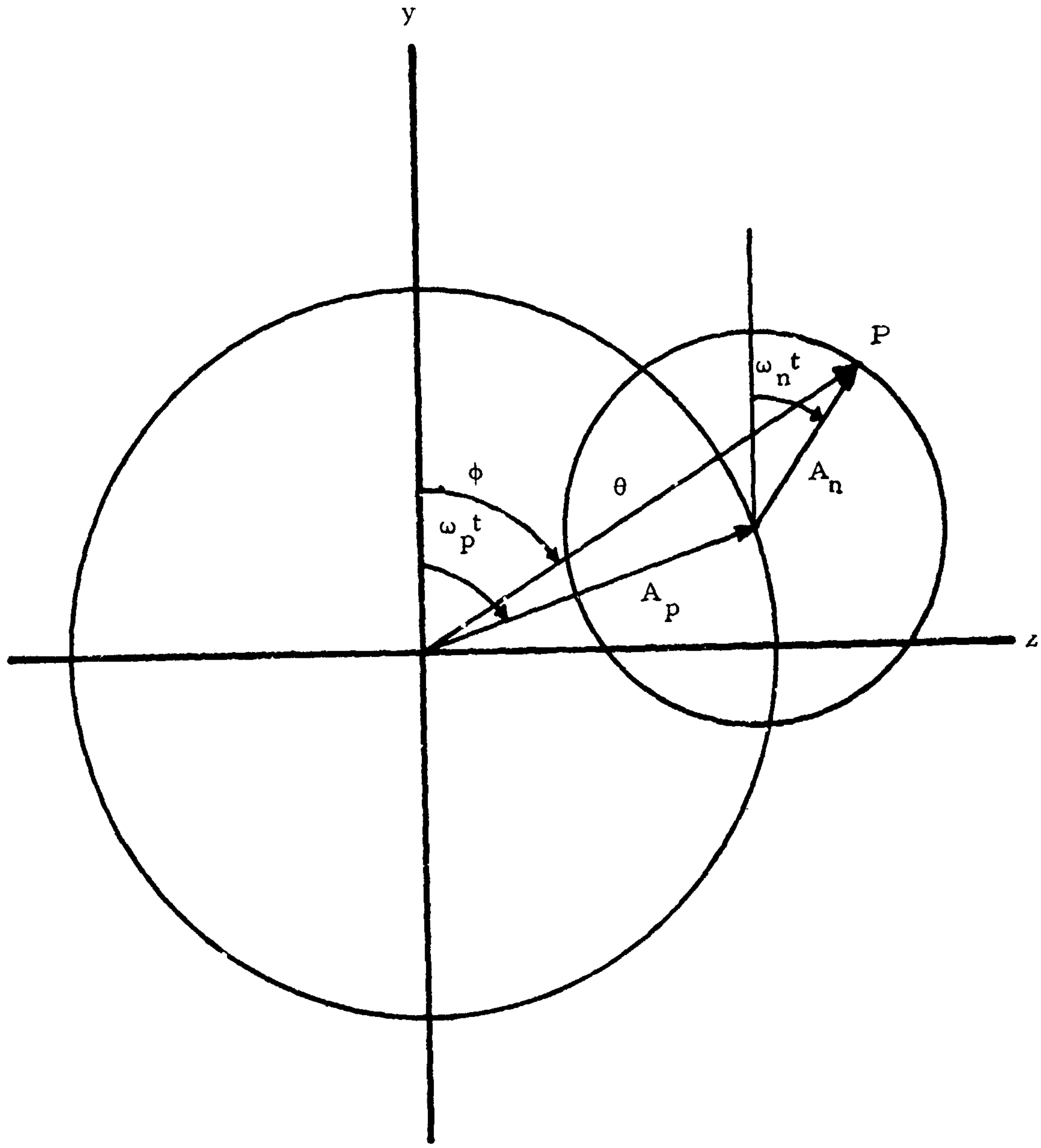


Figure 8

Epicyclic Motion. The x, y, z Axes are the Same as in Fig. 5 with the x Axis into the Page

In terms of complex numbers

$$\theta = \theta_y + i\theta_z = A_p e^{i\omega_p t} + A_n e^{i\omega_n t} = |\theta| e^{i\phi} \tag{100}$$

Subscripts p and n stand for precession and nutation, respectively.

A better approximation to projectile motion is provided if A_p and A_n are damped, that is, if

$$A_p = A_{po} e^{-\mu_p t}$$

$$A_n = A_{no} e^{-\mu_n t}$$

so

$$\theta. = A_{po} e^{-\mu_p t + i\omega_p t} + A_{no} e^{-\mu_n t + i\omega_n t} \tag{101}$$

In this case, the arms A_p and A_n continually get shorter and the motion is similar to that sketched in Fig. 9. This motion is typical of a dynamically stable projectile. The criteria for dynamic stability are

$$\mu_p > 0 \quad \text{and} \quad \mu_n > 0 \tag{102}$$

Damping results from the inclusion of the damping moment

$$\vec{G}_H = - D\vec{\omega}_T \tag{103}$$

where D is assumed to be constant and

$$\vec{\omega}_T = \vec{x}_2\omega_2 + \vec{x}_3\omega_3 \tag{104}$$

The differential equations for $\dot{\omega}_2$ and $\dot{\omega}_3$ with the overturning and damping moments included are

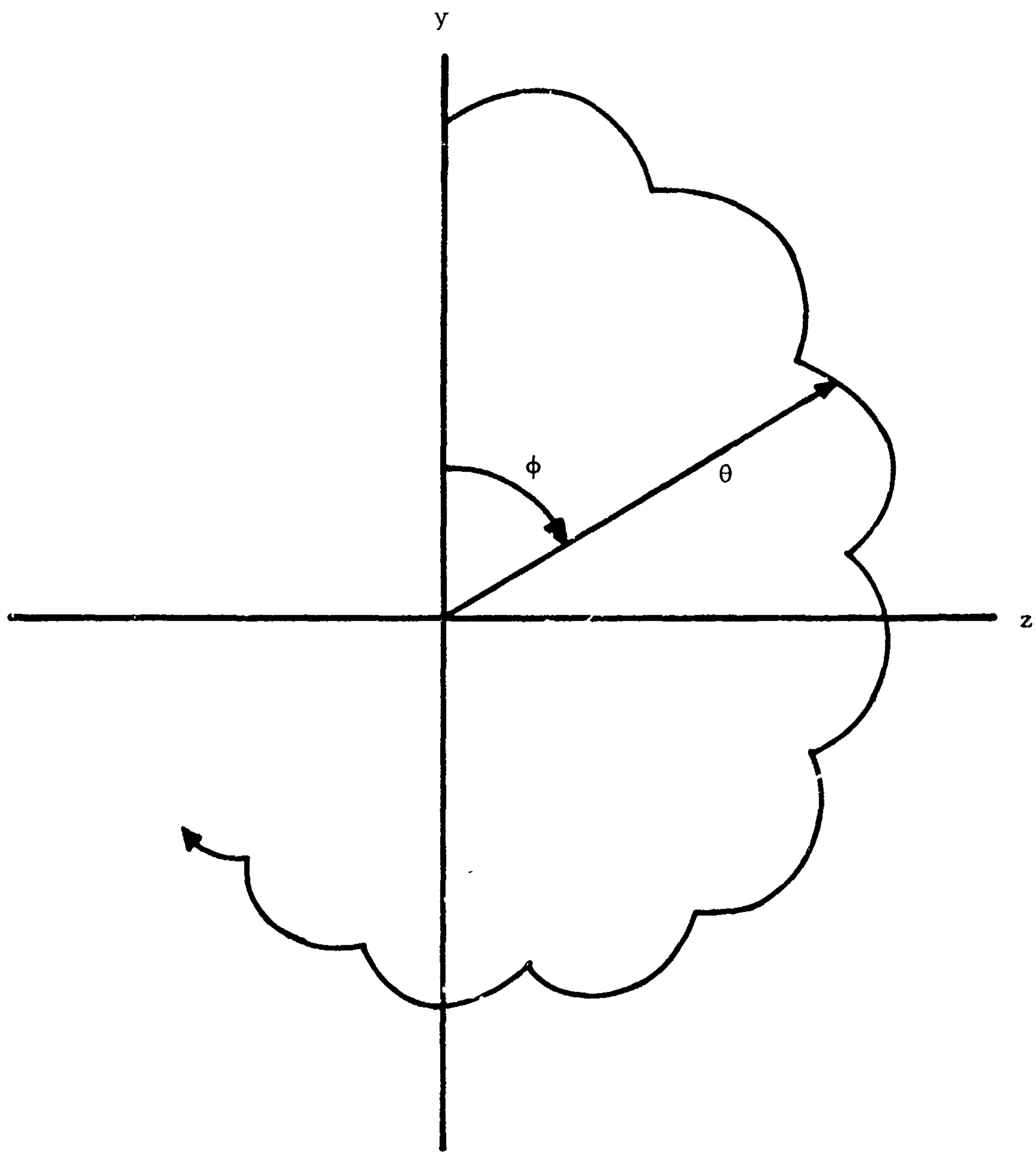


Figure 9
Damped Epicyclic Motion

$$B\dot{\omega}_2 + (A\omega_1 - B\Omega_1)\,\omega_3 = -D\omega_2 \tag{105}$$

$$B\dot{\omega}_3 - (A\omega_1 - B\Omega_1)\,\omega_2 = -D\omega_3 + M \sin\theta \tag{106}$$

Obviously, the damping moment opposes and tends to reduce the angular motion.

6. Complex Notation

The equations of motion can be written in terms of the complex notation of Section II, and an approximate solution can be obtained (Refs. 1, 2, 3, and 16, for example). As in Section II

$$\lambda = \frac{u_2 + iu_3}{u}$$

$$\mu = \frac{(\omega_2 + i\omega_3)d}{u}$$

Time derivatives are transformed into derivatives with respect to arc length, s, measured in calibers, where

$$s = \int_0^t \frac{u}{d}\,dt \tag{107}$$

Differentiation with respect to s will be denoted by a prime, e.g.,

$$u_1' = \frac{du_1}{ds} \tag{108}$$

Upon utilization of Eqs. (44) through (47), and Eqs (62) through (67), the equations of motion in complex notation become

$$\frac{u_1'}{u} + i\,\frac{\mu\bar{\lambda} - \bar{\mu}\lambda}{2} = -\frac{\rho d^3}{m} K_{DA} + \frac{g_1 d}{u^2} \tag{109}$$

$$\lambda' + \left[\frac{u'}{u} + i\,\frac{\Omega_1 d}{u}\right]\lambda - i\,\frac{u_1}{u}\mu$$

$$= \frac{\rho d^3}{m}\left[-K_N + i\nu K_F\right]\lambda + \frac{(g_2 + i g_3)d}{u^2} \tag{110}$$

$$\nu' + \frac{u'}{u}\nu = -\frac{\rho d^3}{m}\,\frac{md^2}{A}\,\nu K_A \tag{111}$$

$$\mu' + \left[\frac{u'}{u} - i\,\frac{A}{B}\nu + i\,\frac{\Omega_1 d}{u}\right]\mu$$

$$= \frac{\rho d^3}{m}\,\frac{md^2}{B}\left[\left\{-\nu K_T - iK_M\right\}\lambda - K_H\mu\right] \tag{112}$$

Aerodynamic forces and moments which are considered negligible have been deleted.

It is convenient to utilize the equation of translational motion which corresponds to the direction of $\vec{u}$. It is

$$m\dot{u} = -\rho d^2 u^2 K_D + mg_u \tag{113}$$

where g_u is the component of $\vec{g}$ in the direction of $\vec{u}$. With the substitution

$$\dot{u} = \frac{u}{d}\cdot u' \tag{114}$$

Equation (113) becomes

$$\frac{u'}{u} = -\frac{\rho d^3}{m}K_D + \frac{g_u d}{u^2} \tag{115}$$

This equation can be used to eliminate u'/u from the equations for λ' and μ'. Upon making the substitutions

$$\ell = \frac{u_1}{u} = \cos\delta \tag{116}$$

and

$$J_X = \frac{\rho d^3}{m} K_X \tag{117}$$

in which X is any of the subscripts D, N, T, etc., Eqs. (110) and (112) become

$$\lambda' + \left[J_N - J_D - i\nu J_F + \frac{g_u d}{u^2} + i\,\frac{\Omega_1 d}{u}\right]\lambda - i\ell\mu = \frac{(g_2 + ig_3)d}{u^2} \tag{118}$$

$$\mu' + \frac{md^2}{B}\left[\nu J_T + i\,J_M\right]\lambda + \left[\frac{md^2}{B} J_H - J_D + \frac{g_u d}{u^2} - i\,\frac{A}{B}\nu + i\,\frac{\Omega_1 d}{u}\right]\mu = 0 \tag{119}$$

Also, the equation for ν' becomes

$$\nu' = -\left[\frac{md^2}{A} J_A - J_D + \frac{g_u d}{u^2}\right]\nu \tag{120}$$

7. Magnitude of Aerodynamic Forces and Moments

It is of interest to estimate the magnitude of the aerodynamic forces and moments to acquire a feel for their effect upon projectile motion. Accordingly, the aerodynamics of the 20-mm, M56 round will be used (Refs. 40, and 52 through 54). For the 20-mm round,

$$M = 0.22 \text{ lb}$$

$$A = 0.00013 \text{ lb ft}^2$$

$$B = 0.00096 \text{ lb ft}^2$$

$$d = 0.06562 \text{ ft}$$

If firings are made at sea level,

$$\rho = 0.076475 \text{ lb/ft}^3$$

and parameters needed in the equations of Subsection 6 are

$$\frac{\rho d^3}{m} = 0.982 \times 10^{-4}$$

$$\frac{md^2}{A} = 7.28$$

$$\frac{md^2}{B} = 0.986$$

$$\frac{A}{B} = 0.135$$

Aerodynamic data at M = 3 for small yaw is

$$K_D \simeq 0.14$$

$$K_N \simeq 1.2$$

$$K_F \simeq 0.1$$

$$K_M \simeq 0.7$$

$$K_H \simeq 2.0$$

$$K_T \simeq -0.05$$

$$K_A \simeq 0.01$$

Except for K_F and K_A, these values were taken from Ref. 53. The value for K_F was calculated from the curve fits in Ref. 54. No data is readily available for K_A, so a more or less typical value is used. The value for K_H is the value obtained when the ball rotor fuze (Refs. 52 and 53) is removed. The measured value with the ball rotor in place, at M = 3, is $K_H \simeq 0.05$.

The value of projectile velocity corresponding to M = 3 at sea level is

$$u = 3350 \text{ ft/sec}$$

A value for ν at the gun muzzle can be obtained from a knowledge of the projectile muzzle velocity, V_M, and the twist of the rifling, n, where

$$V_M = 3300 \text{ ft/sec}$$

$$n = 25.586 \text{ calibers per turn}$$

(Ref. 54). The projectile turns through one revolution when it travels a distance nd in the barrel, so

$$\frac{\omega_1}{V_M} = \frac{2\pi}{nd} \tag{121}$$

and the initial value of ω_1 is

$$\omega_1 = 12,400 \text{ rad/sec}$$

The initial value of ν is

$$\nu = \frac{\omega_1 d}{u} = 0.24$$

The terms due to gravity are less in absolute magnitude than

$$\frac{gd}{u^2} = 1.88 \times 10^{-7}$$

while values of J_X or νJ_X are

$$J_D \simeq 1.4 \times 10^{-5}$$

$$J_N \simeq 1.2 \times 10^{-4}$$

$$\nu J_F \simeq 2.4 \times 10^{-6}$$

$$J_M \simeq 7 \times 10^{-5}$$

$$J_H \simeq 2 \times 10^{-4}$$

$$\nu J_T \simeq -1.2 \times 10^{-6}$$

$$J_A \simeq 10^{-6}$$

Using these typical values, it is seen from Eq. (120) that ν' is small:

$$\frac{md^2}{A} J_A - J_D + \frac{g_u d}{u^2} \simeq 0.7 \times 10^{-5}$$

The solution to

$$\frac{\nu'}{\nu} = -0.7 \times 10^{-5}$$

should be valid at least for small values of s and is

$$\nu = \nu_o e^{-0.7 \times 10^{-5} s}$$

It is clear that ν does not change much over a trajectory.

In the equation for λ', Eq. (118), the leading terms in the coefficient of λ are

$$J_N - i\nu J_F + i\frac{\Omega_1 d}{u}$$

J_D is about 10% of J_N and the gravity term is completely negligible; $i\nu J_F$ is retained because real and imaginary terms are treated separately. The leading terms in the coefficient of μ in the μ' equation, Eq. (119), are

$$\frac{md^2}{B} J_H - i\frac{A}{B}\nu + i\frac{\Omega_1 d}{u}$$

The gravity term is negligible, and J_D is small compared to the damping term containing J_H. The set of equations

$$\lambda' + \left[J_N - i\nu J_F + i\frac{\Omega_1 d}{u}\right]\lambda - i\ell\mu = \frac{(g_2 + ig_3)d}{u^2} \tag{122}$$

$$\mu' + \frac{md^2}{B}\left[\nu J_T + iJ_M\right]\lambda + \left[\frac{md^2}{B} J_H - i\frac{A}{B}\nu + i\frac{\Omega_1 d}{u}\right]\mu = 0 \tag{123}$$

with ν constant as a candidate for approximate solution.

8. An Approximate Solution

To solve Eqs. (122) and (123) with ν constant, a complete specification of the rotating x_1, x_2, x_3 coordinate system must be made so that Ω_1 may be defined (see Subsection 2). This can be done in a number of ways, but for present purposes, it is convenient to define the x_1, x_2, x_3 system so $\Omega_1 = 0$. The details are not required, however, since only a partial solution of projectile motion is desired. Approximate equations for λ and μ will be derived, whereas the development of equations for the direction cosines relating the x_1, x_2, x_3 coordinate to inertial space will be omitted.

Since the influence of gravity over a short distance s along a trajectory is slight, the term

$$\frac{(g_2 + ig_3)d}{u^2}$$

will be omitted.

Also,

$$\ell = \cos \delta$$

is approximately equal to 1 for reasonably small yaw angles, e.g., $\cos \delta = 0.978$ at $\delta = 12^{\circ}$, and so we set

$$\ell = 1$$

The rest of the coefficients are slowly varying and it will be assumed that they are constant. The solution thus obtained will not be exact, but it will show the character of the true solution. The equations to be solved may be written

$$\lambda' + a_1 \lambda + a_2 \mu = 0 \tag{124}$$

$$\mu' + b_1 \lambda + b_2 \mu = 0 \tag{125}$$

where

$$a_1 = J_N - i \nu J_F \tag{126}$$

$$a_2 = -i \tag{127}$$

$$b_1 = \frac{md^2}{B} \left[\nu J_T + i J_M \right] \tag{128}$$

$$b_2 = \frac{md^2}{B} J_H - i \frac{A}{B} \nu \tag{129}$$

The solution is straightforward (see Ref. 55). Set

$$\lambda = \lambda_o e^{ks} \tag{130}$$

$$\mu = \mu_o e^{ks} \tag{131}$$

where λ_o, μ_o, and k are constants. Substitution of these expressions into the differential equations yields

$$(k + a_1)\lambda_o + a_2\mu_o = 0 \tag{132}$$

$$b_1\lambda_o + (k + b_2)\mu_o = 0 \tag{133}$$

and nonzero solutions exist for λ_o and μ_o only if the determinant

$$\begin{vmatrix} k + a_1 & a_2 \\ b_1 & k + b_2 \end{vmatrix} \tag{134}$$

equals zero or if

$$k^2 + (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0 \tag{135}$$

The solution of this quadratic equation is

$$k = \frac{-(a_1 + b_2) \pm \sqrt{(a_1 + b_2)^2 - 4(a_1b_2 - a_2b_1)}}{2} \tag{136}$$

Now

$$a_1 + b_2 = J_N - i\nu J_F + \frac{md^2}{B} J_H - i\frac{A}{B}\nu$$

$$\simeq J_N + \frac{md^2}{B} J_H - i\frac{A}{B}\nu \tag{137}$$

and

$$(a_1 + b_2)^2 - 4(a_1 b_2 - a_2 b_1) = (a_1 - b_2)^2 + 4a_2 b_1$$

$$= \left[J_N - \frac{md^2}{B} J_H + i\frac{A}{B}\nu\right]^2 - 4i\frac{md^2}{B}\left[\nu J_T + iJ_M\right]$$

$$\simeq \left\{4\frac{md^2}{B} J_M - (\frac{A}{B}\nu)^2\right\}$$

$$+ 2i\nu\left\{\frac{A}{B} J_N - \frac{A}{B}\frac{md^2}{B} J_H - 2\frac{md^2}{B} J_T\right\} \tag{138}$$

where negligible terms have been deleted. With

$$a = J_N + \frac{md^2}{B} J_H \tag{139}$$

$$b = \frac{A}{B}\nu \tag{140}$$

$$c = 4\frac{md^2}{B} J_M - \left(\frac{A}{B}\nu\right)^2 \tag{141}$$

$$e = 2\nu\left\{\frac{A}{B} J_N - \frac{A}{B}\frac{md^2}{B} J_H - 2\frac{md^2}{B} J_T\right\} \tag{142}$$

we have

$$2k = -a + ib \pm \sqrt{c + ie} \tag{143}$$

According to DeMoivre's theorem (Ref. 56)

$$\sqrt{c + ie} = \sqrt[4]{c^2 + e^2} \left[\cos\frac{\alpha}{2} + i \sin\frac{\alpha}{2}\right] \tag{144}$$

where

$$\cos\alpha = \frac{c}{\sqrt{c^2 + e^2}} \tag{145}$$

$$\sin\alpha = \frac{e}{\sqrt{c^2 + e^2}} \tag{146}$$

But

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1 + \cos\alpha}{2}} \tag{147}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1 - \cos\alpha}{2}} \tag{148}$$

and so

$$\sqrt{c + ie} = \sqrt{\frac{1}{2}\left(\sqrt{c^2 + e^2} + c\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{c^2 + e^2} - c\right)} = h + iq \tag{149}$$

(It has been assumed that α is in the first quadrant; if not, appropriate changes must be made in the signs of the radicals in Eqs. (147) and (148).) It follows that

$$2k = -a + ib \pm (h + iq) \tag{150}$$

and that

$$\lambda = \lambda_p e^{-\alpha_p s + i\beta_p s} + \lambda_n e^{-\alpha_n s + i\beta_n s} \tag{151}$$

where λ_p and λ_n are constants, and

$$\alpha_p = \frac{1}{2}(a - h) \tag{152}$$

$$\alpha_n = \frac{1}{2}(a + h) \tag{153}$$

$$\beta_p = \frac{1}{2}(b + q) \tag{154}$$

$$\beta_n = \frac{1}{2}(b - q) \tag{155}$$

A similar expression exists for μ, and it is seen that the motions of λ and μ are epicyclic in the variable s. For dynamic stability, it is required that

$$\alpha_p > 0 \quad \text{and} \quad \alpha_n > 0 \tag{156}$$

It is observed that $|\lambda| = \sin\delta$, where δ is the angle between the projectile spin axis and the velocity vector, and the precessional and nutational motion is about the velocity vector. The plane of yaw is the plane containing the spin axis and the velocity vector and so it precesses with the projectile. Since the lift force is in the plane of yaw, and the Magnus force is perpendicular to it, these forces also precess about the velocity vector.

The analysis of this section is approximate, and the results are tutorial. A better approximation is given in Ref. 16 along with a development of criteria for static and dynamic stability. A treatment similar to that of Ref. 16, with slightly less general results, may be found in Ref. 1. In passing, it is observed without proof that the condition for gyroscopic stability is

$$s^* > 1 \tag{157}$$

where

$$s^* = \frac{(A\nu/B)^2}{4(md^2/B)J_M} = \frac{A^2\omega_1^2}{4B\rho d^3u^2K_M} \tag{158}$$

(Refs. 1 and 16).

9. Windage Jump and Drift

Windage jump and drift may both be interpreted as deflections of the projectile trajectory away from its initial direction of motion due to aerodynamic forces. Windage jump is caused by the precessional and nutational motion of the projectile near the gun muzzle, while drift results from the effects of gravity at long ranges. Whereas drift has not been found important in the past, it may be significant for the new, heavy, high-muzzle-velocity rounds under development.

The windage jump arises from the side forces on the projectile and the precessional motion of the projectile about the velocity vector. The side forces are the lift force in the plane of yaw, and the Magnus force perpendicular to the plane of yaw. These forces would be zero if the angle of attack, δ, were zero. As the spinning projectile moves down its trajectory, the aerodynamic moments cause it to precess about its direction of motion like a top under gravity and so the plane of yaw precesses with the projectile. The side forces are carried with the plane of yaw and the changing direction of the forces moves the projectile first in one direction and then in another. For a dynamically stable projectile, the angle of attack decreases (on the average) with time and the side forces decrease. The center of mass of the projectile moves downrange along a "spiral" of continuously decreasing radius. The angle of attack decreases to effectively zero usually within the first 1000 ft and the spiraling motion stops. The net result is that the direction of motion of the projectile is changed by the amount of the windage jump. Equations for the windage jump are given in Sections IV and V. An explanation of the drift is as follows.

At long ranges, and in the absence of gravity, the yawing motion of the projectile would completely die out. Gravity, however, causes curvature in the trajectory and the continually changing direction of the velocity vector induces a yaw angle, called the yaw of repose, between the projectile spin axis and the velocity vector. A balance of aerodynamic and gyroscopic moments causes the projectile to move with its nose pointed slightly up and to the right of the trajectory, and the effect is to deflect the trajectory up and to the right. A treatment of drift is given in Ref. 57.

SECTION IV

EQUATIONS OF MOTION FOR COMPUTER UTILIZATION

1. General

The complex number representation of the six-degree-of-freedom equations of motion derived in Section III is useful for analytical studies of the behavior of a projectile in flight, but it is not suitable for rapid computation on a digital computer. Two different formulations of the equations of motion are presented in this section. The first, a matrix formulation, can be advantageously programmed on a digital computer for the rapid generation of trajectory tables, whereas the second method, an Euler angle approach, is more adaptable to numerical studies and to approximation. Derivations for the matrix formulation and the Euler angle representation follow in Subsections 2 and 3. In Subsection 4, approximate equations are developed from the Euler angle set for the approximate computation of trajectory tables. These equations are useful when it is permissible to sacrifice accuracy in favor of speed in computation. Initial conditions are not given in this section. For computer studies, these parameters may be chosen arbitrarily, whereas they must be calculated in airborne applications. Calculation of initial conditions in airborne applications is treated in Section VI.

2. A Matrix Formulation

A right-handed X, Y, Z inertial coordinate system is defined as follows: X and Y are horizontal and Z is vertical. The X, Y, and Z coordinates refer to the center of mass of the projectile and unit vectors $\vec{X}$, $\vec{Y}$, and $\vec{Z}$ are defined in the indicated directions.

A moving x_1, x_2, x_3 system is defined in Section II with its origin at the center of mass of the projectile. The x_1 axis is along the longitudinal axis of the projectile and is positive toward the nose. The x_2 and x_3 axes are normal to x_1 but are not fixed in the projectile. The x_1, x_2, x_3 system is right-handed and $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are unit vectors in the indicated directions. The orientation of the x_1, x_2, x_3 system with respect to the X, Y, Z system is given by the relation

$$\begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \end{pmatrix} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{pmatrix} \tag{159}$$

The angular velocity of the x_1, x_2, x_3 system is

$$\vec{\Omega} = \vec{x}_1\Omega_1 + \vec{x}_2\Omega_2 + \vec{x}_3\Omega_3$$

while the linear and angular velocities of the projectile are

$$\vec{u} = \vec{x}_1 u_1 + \vec{x}_2 u_2 + \vec{x}_3 u_3$$

and

$$\vec{\omega} = \vec{x}_1\omega_1 + \vec{x}_2\omega_2 + \vec{x}_3\omega_3$$

The angular momentum of the shell is

$$\vec{H} = \vec{x}_1 A\omega_1 + \vec{x}_2 B\omega_2 + \vec{x}_3 B\omega_3$$

where A and B are, respectively, the axial and transverse moments of inertia of the projectile. The equations of motion are

$$\vec{F} = m\dot{\vec{u}} = m(\vec{\dot{u}} + \vec{\Omega} \times \vec{u})$$

and

$$\vec{G} = \dot{\vec{H}} = \vec{\dot{H}} + \vec{\Omega} \times \vec{H}$$

A dot over the arrow refers to a time derivative measured in the X, Y, Z system, whereas a dot under the arrow denotes a time derivative measured in the moving system. $\vec{F}$ is force, $\vec{G}$ is torque, and m is the projectile mass. The equations of motion in matrix form are

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = m \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} + m \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{160}$$

and

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{161}$$

The restriction that the x_1 axis coincides with the projectile longitudinal axis implies that

$$\Omega_2 = \omega_2 \tag{162}$$

$$\Omega_3 = \omega_3 \tag{163}$$

Ω_1 may be chosen arbitrarily. The choice $\Omega_1 = \omega_1$ is inconvenient for numerical integration because a very small step size (of the order of

$0.1 \times 2\pi/\omega_1$) will be required. Also, we have no interest in the angular orientation of the projectile about the x_1 axis. A second choice is to constrain the x_1, x_2, x_3 coordinate system to move in a manner such that the x_3 axis remains in the horizontal plane. This choice is convenient since x_2 is always in a vertical plane and Ω_1 is small. A third convenient choice is $\Omega_1 = 0$.

Forces and moments are

$$F_1 = -\rho d^2 u^2 K_{DA} \qquad - mg\ell_3 \tag{164}$$

$$F_2 = -\rho d^2 u K_N u_2 - \rho d^2 u \nu K_F u_3 \quad - mgm_3 \tag{165}$$

$$F_3 = -\rho d^2 u K_N u_3 + \rho d^2 u \nu K_F u_2 - mg n_3 \tag{166}$$

$$G_1 = -\rho d^3 u^2 \nu K_A \tag{167}$$

$$G_2 = -\rho d^3 u \nu K_T u_2 + \rho d^3 u K_M u_3 - \rho d^4 u K_H \omega_2 \tag{168}$$

$$G_3 = -\rho d^3 u \nu K_T u_3 - \rho d^3 u K_M u_2 - \rho d^4 u K_H \omega_3 \tag{169}$$

where g is the acceleration due to gravity. The aerodynamic forces and moments are the same as those of Section II, Subsection 5 with negligible terms deleted. If aerodynamic data for K_D and K_L is available rather than that for K_{DA} and K_N, the following expressions may be used

$$K_{DA} = K_D \cos\delta - K_L \sin^2\delta \tag{170}$$

$$K_N = K_L \cos\delta + K_D \tag{171}$$

Also

$$\cos\delta = \frac{u_1}{u} \tag{172}$$

$$\nu = \frac{\omega_1 d}{u} \tag{173}$$

$$u = \sqrt{u_1^2 + u_2^2 + u_3^2} \tag{174}$$

The aerodynamic coefficients (the K's) are functions of Mach number, sin δ, ν, and possibly Reynolds number (see Section II).

The matrix equation for the velocity is

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{175}$$

and the matrix equation for the time rates of cnange of the direction cosines is

$$\frac{d}{dt} \begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix} = \begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix} \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \tag{176}$$

This set of matrix equations, along with the equations for the aerodynamic forces and moments, completely describes the motion of a projectile. For numerical solution, it has the advantage that no trigonometric functions need be evaluated from angles ($\cos\delta = u_1/u$) and as a consequence

computations are fast. It also has the advantage that no division by small numbers occurs, such as division by sin δ for small δ. The problem of division by small numbers often occurs in formulations involving Euler's angles. But, instead of the expected twelve equations, there are eighteen! The redundancy is contained in the equation for the direction cosines. One might use the properties of the direction cosine matrix (orthogonality, etc.) to reduce the number of equations, but this has not been found to be advantageous. Instead, all eighteen equations are integrated simultaneously. (Actually only seventeen need be integrated since the angle of rotation of the projectile about its spin axis is of no interest.) Numerical problems occur which are associated with the direction cosine matrix not remaining orthogonal, however. One method of correcting this deficiency is to replace the matrix by

$$\frac{1}{2}\left(\Lambda + \Lambda_T^{-1}\right)$$

where Λ is the direction cosine matrix and Λ_T^{-1} is the inverse of the transpose (Ref. 58).

3. Euler Angle Development

Large portions of the development given here and in the next subsection is taken almost verbatim from Ref. 59. Changes have been made as necessary, however, to clarify and adapt Ref. 59 to present purposes. Portions not pertinent to present needs have been deleted.

A right-handed, orthogonal, rectilinear ξ, η, ζ coordinate system which is stationary with respect to the ground is defined with ξ measured down range, η vertically up, and ζ to the right as seen by a person facing down range. A second right-handed, orthogonal, rectilinear coordinate system I', J', K' is superimposed on ξ, η, ζ. The two coordinate systems have a common origin O, and the angular orientation of I', J', K' with respect to ξ, η, ζ is specified by the two angles α and θ (see Fig. 10). The angles α and θ are by definition the azimuth and elevation angles, respectively, of the bullet velocity

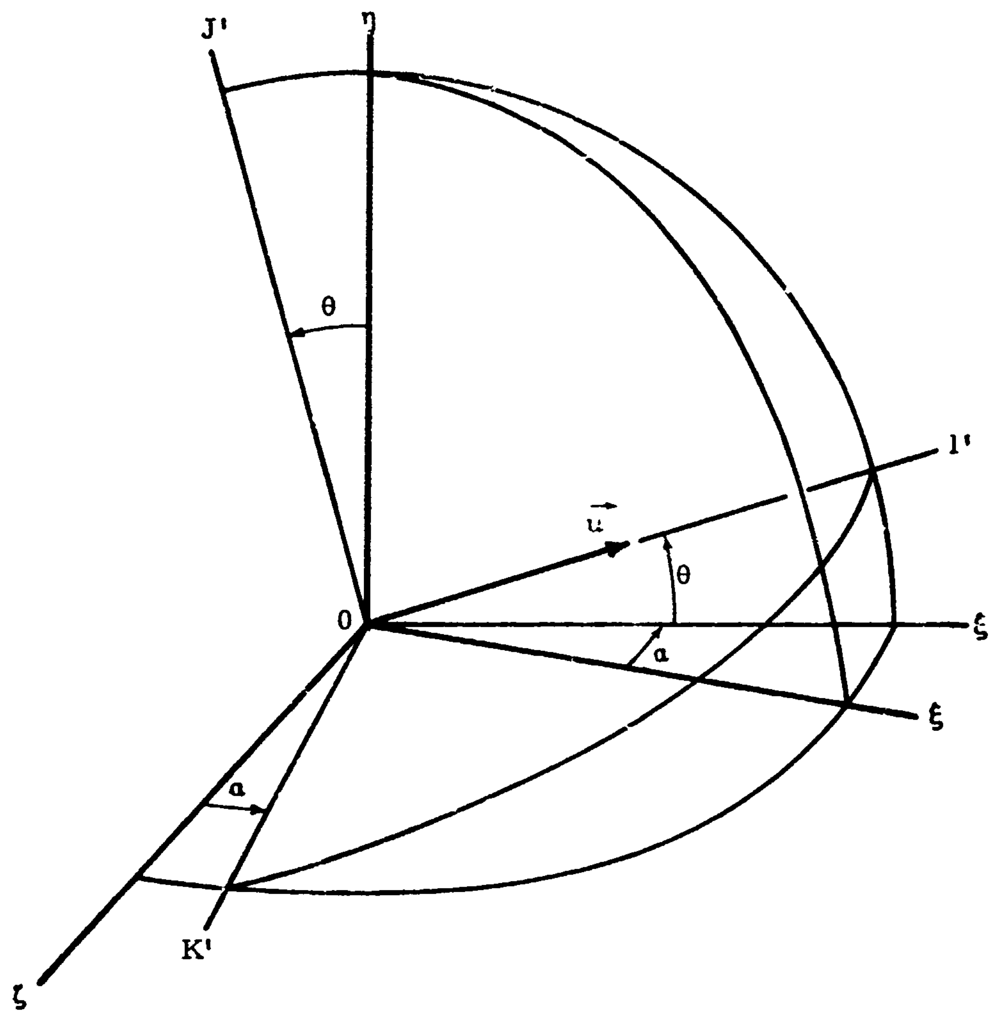


Figure 10
Velocity Vector Coordinates

vector $\vec{u}$ as measured in the ξ, η, ζ coordinate system, and 1' is always along $\vec{u}$. The azimuth angle α is measured from the ξ axis to ξ', the projection of $\vec{u}$ on the horizontal plane, and the elevation angle θ is measured from ξ' to $\vec{u}$ or to 1'. The angle α is a counterclockwise rotation as seen from above (from the positive η axis) and θ is a counterclockwise rotation about K' as seen from the positive K' axis. Note that K' and ξ' are coplanar with ζ and ξ and are horizontal, and that 1' and J' are coplanar with ξ' and η.

Now, define a 1, J, K coordinate system parallel to 1', J', K' but moving with the projectile and with its origin at the center of mass of the projectile. In this system, define a right-handed, orthogonal coordinate system 1, 2, 3, (see Fig. 11). The 1-2 plane contains the axis of symmetry, A, of the projectile and is the plane of yaw; A is directed from the projectile center of mass toward the nose. The angle ϕ is a counterclockwise rotation about 1 as seen from 1 and is measured in the J-K plane from J to 2. The angle δ, the yaw angle, is the angle in the plane of yaw between the projectile velocity vector and the axis of symmetry, and it is a counterclockwise rotation about 3 as seen from 3. The angle ψ (not shown in the figure) is the roll angle of the projectile measured about A, and it is assumed to be a counterclockwise rotation as seen from A; $\dot{\psi}$ is the roll rate of the projectile.

It is observed that the angles δ, ϕ, and ψ, which define the projectile orientation with respect to the moving 1, J, K system, are Euler angles. Contrary to the notation of Section II, the 1, 2, 3, system is attached to $\vec{u}$ rather than to the projectile. The A, B, 3 coordinates used here correspond to the x_1, x_2, x_3 system defined in Section II.

Figures 10 and 11 show that the orientation angles of the 1, 2, 3 coordinate system are α, θ, and ϕ, and that 1, 2, 3, constitutes a coordinate system with one axis parallel to the velocity vector $\vec{u}$ and with the other two axes rotating at essentially the precession rate of the projectile. The angular velocity of the 1, 2, 3 coordinate system with respect to the fixed system is

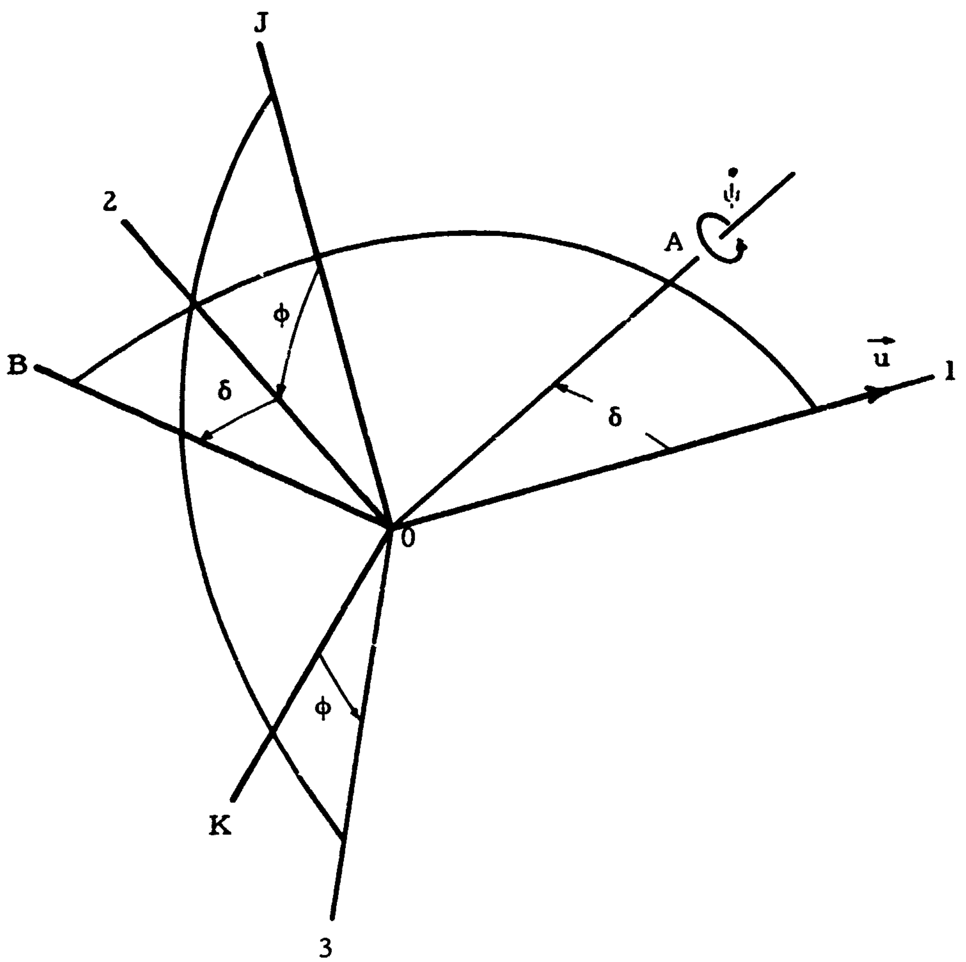


Figure 11
Coordinates of Angular Motion

$$\Omega = \vec{1}_\eta \dot{\alpha} + \vec{1}_K \dot{\theta} + \vec{1}_1 \dot{\phi}$$

where $\vec{1}_\eta$, $\vec{1}_K$, and $\vec{1}_1$ are unit vectors along η, K(or K') and 1, respectively, and a dot represents differentiation with respect to time. To obtain components of $\vec{\Omega}$ along the 1, 2, and 3 directions, unit vectors $\vec{1}_\eta$ and $\vec{1}_K$ are resolved along the 1 and 2 directions as follows:

$$\vec{1}_K = \vec{1}_2 \sin\phi + \vec{1}_3 \cos\phi$$

$$\vec{1}_\eta = \vec{1}_1 \sin\theta + \vec{1}_J \cos\theta$$

$$= \vec{1}_1 \sin\theta + (\vec{1}_2 \cos\phi - \vec{1}_3 \sin\phi) \cos\theta$$

The equations for $\vec{1}_J$ and $\vec{1}_K$ are obtained directly from Fig. 11, but $\vec{1}_\eta$ is obtained from Fig. 10. Substitution of $\vec{1}_K$ and $\vec{1}_\eta$ into the expression for $\vec{\Omega}$ yields the 1, 2, 3 components of $\vec{\Omega}$ and they are

$$\Omega_1 = \dot{\phi} + \dot{\alpha} \sin\theta \tag{177}$$

$$\Omega_2 = \dot{\alpha} \cos\phi \cos\theta + \dot{\theta} \sin\phi \tag{178}$$

$$\Omega_3 = -\dot{\alpha} \sin\phi \cos\theta + \dot{\theta} \cos\phi \tag{179}$$

Note that $\dot{\alpha}$ and $\dot{\theta}$ are small since the direction of the velocity vector does not change much, and consequently Ω_2 and Ω_3 are small. Ω_1, therefore, is essentially the precession rate of the projectile.

Figure 11 shows that the angular velocity of the projectile is

$$\vec{\omega} = \vec{\Omega} + \vec{1}_A \dot{\psi} + \vec{1}_3 \dot{\delta}$$

and therefore, the components of angular velocity of the projectile resolved along the A, B, 3 axes are

$$\omega_A = \Omega_1 \cos\delta + \Omega_2 \sin\delta + \dot{\psi} = \Omega_A + \dot{\psi} \tag{180}$$

$$\omega_B = -\Omega_1 \sin\delta + \Omega_2 \cos\delta = \Omega_B \tag{181}$$

$$\omega_3 = \Omega_3 + \dot{\delta} \tag{182}$$

Components of the angular momentum of the projectile are

$$H_A = A\omega_A \tag{183}$$

$$H_B = B\omega_B \tag{184}$$

$$H_3 = B\omega_3 \tag{185}$$

where A and B are the axial and transverse moments of inertia of the projectile. It is assumed that the projectile is symmetrical about its longitudinal axis, and that the moments of inertia about all transverse axes through the center of mass are equal.

Newton's second law for rotational motion can be written as

$$\vec{G} = \dot{\vec{H}} + \vec{\Omega}' \times \vec{H} \tag{186}$$

where $\vec{G}$ is the aerodynamic torque, and

$$\vec{\Omega}' = \vec{\omega} - \vec{1}_A \dot{\psi}$$

is the angular velocity of the A, B, 3 coordinate system; that is, $\vec{\Omega}'$ is equal to $\vec{\omega}$ minus the axial spin. Components of $\vec{\Omega}'$ along A, B, and 3 are

$$\Omega'_A = \omega_A - \dot{\psi} = \Omega_A$$

$$\Omega'_B = \omega_B$$

$$\Omega'_3 = \omega_3$$

and corresponding components of Eq. (186) are

$$G_A = A\dot{\omega}_A \tag{187}$$

$$G_B = B\dot{\omega}_B + \omega_3 A\omega_A - \Omega_A B\omega_3 \tag{188}$$

$$G_3 = B\dot{\omega}_3 + \Omega_A B\omega_B - \Omega_B A\omega_A \tag{189}$$

Equations (187), (188), and (189) constitute the angular motion equations of the projectile, and G_A, G_B, and G_3 are the aerodynamic moments.

Newton's second law for linear motion with velocity and force components resolved along the 1, 2, 3 directions is

$$\vec{F} = m\left[\dot{\vec{u}} + \vec{\Omega} \times \vec{u}\right] \tag{190}$$

and also

$$F_1 = m\dot{u} \tag{191}$$

$$F_2 = m\Omega_3 u \tag{192}$$

$$F_3 = -m\Omega_2 u \tag{193}$$

The components of $\vec{F}$ and $\vec{G}$ may be identified with the proper aerodynamic forces and moments, from Fig. 11 and the results of Section II, with careful attention to geometry. The x_1, x_2, x_3 directions of Section II may be identified with the A, B, 3 directions, respectively, in this section, and in Eqs. (44), (48), and (49), F_1 becomes F_A, F_2 becomes F_B, u_2 becomes $u_B = -u \sin \delta$, and u_3 is zero. If gravity is ignored for the moment, and negligible aerodynamic terms are discarded, it follows that

$$F_A = -\rho d^2 u^2 K_{DA} \quad (194)$$

$$F_B = \rho d^2 u^2 K_N \sin\delta \quad (195)$$

$$F_3 = -\rho d^2 u^2 \nu K_F \sin\delta \quad (196)$$

But, for the 1 and 2 directions of this subsection

$$F_1 = F_A \cos\delta - F_B \sin\delta = -\rho d^2 u^2 (K_{DA} \cos\delta + K_N \sin^2\delta) \quad (197)$$

$$F_2 = F_A \sin\delta + F_B \cos\delta = \rho d^2 u^2 (-K_{DA} + K_N \cos\delta) \sin\delta \quad (198)$$

From Eqs. (54) and (55), and when the gravity terms are added in, it follows that

$$F_1 = -\rho d^2 u^2 K_D \qquad - mg \sin\theta \quad (199)$$

$$F_2 = \rho d^2 u^2 K_L \sin\delta \quad - mg \cos\theta \cos\phi \quad (200)$$

$$F_3 = -\rho d^2 u^2 \nu K_F \sin\delta + mg \cos\theta \sin\phi \quad (201)$$

In Eqs. (46), (50), and (51) for the aerodynamic torques, identification of the x_1, x_2, x_3 system with the A, B, 3 system implies that G_1 becomes G_A, G_2 becomes G_B, u_2 becomes $u_B = -u \sin\delta$ (as before), u_3 is zero (as before), and ω_2 becomes ω_B. When negligible terms are discarded it follows that

$$G_A = -\rho d^3 u^2 \nu K_A \quad (202)$$

$$G_B = \rho d^3 u^2 \nu K_T \sin\delta - \rho d^4 u K_H \omega_B \quad (203)$$

$$G_3 = \rho d^3 u^2 K_M \sin\delta - \rho d^4 u K_H \omega_3 \quad (204)$$

where

$$\nu = \frac{\omega_A d}{u} \tag{205}$$

Equations for $\dot{\alpha}$ and $\dot{\theta}$ can be obtained in terms of Ω_2 and Ω_3 from Eqs. (178) and (179).

$$\dot{\alpha} \cos\theta = \Omega_2 \cos\phi - \Omega_3 \sin\phi$$

$$\dot{\theta} = \Omega_2 \sin\phi + \Omega_3 \cos\phi$$

From Eqs. (192) and (193) it follows that

$$\dot{\alpha} \cos\theta = \frac{1}{mu} \left[- F_3 \cos\phi - F_2 \sin\phi \right]$$

$$\dot{\theta} = \frac{1}{mu} \left[- F_3 \sin\phi + F_2 \cos\phi \right]$$

and finally, from Eqs. (200) and (201)

$$\dot{\alpha} = \frac{\rho d^2 u}{m \cos\theta} \left[\nu K_F \cos\phi - K_L \sin\phi \right] \sin\delta \tag{206}$$

$$\dot{\theta} = \frac{\rho d^2 u}{m} \left[\nu K_F \sin\phi + K_L \cos\phi \right] \sin\delta - \frac{g}{u} \cos\theta \tag{207}$$

(In numerical integration, division by $\cos\theta$ causes numerical trouble when θ is near $\pm 90^{\circ}$.) From Eqs. (191) and (199)

$$\dot{u} = - \frac{\rho d^2 u^2}{m} K_D - g \sin\theta \tag{208}$$

Equations (206), (207), and (208) are the equations for $\dot{\bar{u}}$ in polar coordinates. The equations for projectile position can be obtained from examination of Fig. 10. They are

$$\dot{\xi} = u \cos\theta \cos\alpha \tag{209}$$

$$\dot{\eta} = u \sin\theta \tag{210}$$

$$\dot{\zeta} = -u \cos\theta \cos\alpha \tag{211}$$

By use of Eqs. (187), (188), (189), (202), (203), and (204), the angular motion equations may now be written as

$$A\dot{\omega}_A = -\rho d^4 u\omega_A K_A \tag{212}$$

$$B\dot{\omega}_B + \omega_3 A\omega_A - \Omega_A B\omega_3$$
$$= \rho d^4 u\omega_A K_T \sin\delta - \rho d^4 u K_H \omega_B \tag{213}$$

$$B\dot{\omega}_3 + \Omega_A B\omega_B - \omega_B A\omega_A$$
$$= \rho d^3 u^2 K_M \sin\delta - \rho d^4 u K_H \omega_3 \tag{214}$$

where ν has been replaced by the right side of Eq. (205). Ω_A is obtained as follows: use Eqs. (178) and (179) to calculate Ω_2 and Ω_3; then Ω_1 may be obtained from Eq. (181), i.e.

$$\Omega_1 = \frac{\Omega_2 \cos\delta - \omega_B}{\sin\delta} \tag{215}$$

(Division by sin δ causes numerical integration problems when δ is small.) Given Ω_1 and Ω_2, Eq. (180) yields Ω_A. Also, $\dot{\phi}$ and $\dot{\delta}$ may be obtained from Eqs. (177) and (182).

3.1 Summary of Equations - It is convenient to collect together the complete set of six-degree-of-freedom equations. They are as follows:

$$\dot{\alpha} = \frac{\rho d^2 u}{m \cos\theta} \left[\nu K_F \cos\phi - K_L \sin\phi \right] \sin\delta$$

$$\dot{\theta} = \frac{\rho d^2 u}{m} \left[\nu K_F \sin\phi + K_L \cos\phi \right] \sin\delta - \frac{g}{u} \cos\theta$$

$$\dot{u} = -\frac{\rho d^2 u^2}{m} K_D - g \sin\theta$$

$$\Omega_2 = \dot{\alpha} \cos\theta \cos\phi + \dot{\theta} \sin\phi$$

$$\Omega_3 = -\dot{\alpha} \cos\theta \sin\phi + \dot{\theta} \cos\phi$$

$$\Omega_1 = \frac{-\omega_B + \Omega_2 \cos\delta}{\sin\delta}$$

$$\Omega_A = \Omega_1 \cos\delta + \Omega_2 \sin\delta$$

$$\dot{\omega}_A = -\frac{\rho d^4 u}{A} \omega_A K_A$$

$$\nu = \frac{\omega_A d}{u}$$

$$\dot{\omega}_B = -\omega_3 \left[\frac{A}{B} \omega_A - \Omega_A \right] + \frac{\rho d^3 u^2}{B} \left[\nu K_T \sin\delta - \frac{\omega_B d}{u} K_H \right]$$

$$\dot{\omega}_3 = \omega_B \left[\frac{A}{B} \omega_A - \Omega_A \right] + \frac{\rho d^3 u^2}{B} \left[K_M \sin\delta - \frac{\omega_3 d}{u} K_H \right]$$

$$\dot{\delta} = \omega_3 - \Omega_3$$

$$\dot{\phi} = \Omega_1 - \dot{\alpha} \sin \theta$$

$$\dot{\xi} = u \cos \theta \cos \alpha$$

$$\dot{\eta} = u \sin \theta$$

$$\dot{\zeta} = -u \cos \theta \sin \alpha$$

Initial conditions are treated in Section VI.

4. Approximate Equations for Large-Yaw Computations

This subsection contains a derivation of the approximate equations of motion used at Eglin Air Force Base for trajectory table calculations for the 20-mm, M56 round. An equivalent set has also been used in the computation of tables by BRL (Ref. 54). Presumably, the method could be adapted to new rounds under development. Another derivation of the two approximate equations of angular motion can be found in Ref. 60. This derivation is quite tedious, however, and derivations of the other equations are lacking. The development herein is that of Ref. 59 with minor changes for clarity and is based upon the equations of the last subsection. The equations of linear motion must be formulated in the ξ, η, ζ system, however.

The reason for using these approximate equations is they can be evaluated much faster on a digital computer than can the six-degree-of-freedom equations. The angular motion is approximated and the nutation motion (Section III) is eliminated. Since the fine detail of the nutational motion is absent, and only the precessional motion is left, the numerical step size can be increased and thereby computer time is reduced. The derivation proceeds as follows.

Equation (192) is combined here with Eq. (200) for future reference:

$$m\Omega_3 u = \rho d^2 u^2 K_L \sin\delta - mg\cos\theta\cos\phi \tag{216}$$

Equations (199), (200), and (201) transformed to the 1, J, K system are

$$F_1 = -\rho d^2 u^2 K_D - mg\sin\theta \tag{217}$$

$$F_J = \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right]\sin\delta - mg\cos\theta \tag{218}$$

$$F_K = \rho d^2 u^2 \left[K_L \sin\phi - \nu K_F \cos\phi\right]\sin\delta \tag{219}$$

And in terms of the ξ, η, ζ system

$$F_\xi = F_1 \cos\theta\cos\alpha - F_J \sin\theta\cos\alpha + F_K \sin\alpha \tag{220}$$

$$F_\eta = F_1 \sin\theta + F_J \cos\theta \tag{221}$$

$$F_\zeta = -F_1 \cos\theta\sin\alpha + F_J \sin\theta\sin\alpha + F_K \cos\alpha \tag{222}$$

If Eqs. (217), (218), and (219) are combined with Eqs. (220), (221), and (222), and use is made of Eqs. (209), (210), and (211), the following expressions are obtained

$$F_\xi = -\rho d^2 u K_D \dot{\xi} - \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right]\sin\delta\sin\theta\cos\alpha$$

$$+\rho d^2 u^2 \left[K_L \sin\phi - \nu K_F \cos\phi\right]\sin\delta\sin\alpha \tag{223}$$

$$F_\eta = -\rho d^2 u K_D \dot{\eta} + \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right]\sin\delta\cos\theta - mg \tag{224}$$

$$F_\zeta = -\rho d^2 u K_D \dot{\zeta} + \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right]\sin\delta\sin\theta\sin\alpha$$

$$+\rho d^2 u^2 \left[K_L \sin\phi - \nu K_F \cos\phi\right]\sin\delta\cos\alpha \tag{225}$$

Finally,

$$m\ddot{\xi} = F_{\xi} \tag{226}$$

$$m\ddot{\eta} = F_{\eta} \tag{227}$$

$$m\ddot{\zeta} = F_{\zeta} \tag{228}$$

4.1 The Siacci-Type Approximation to the Force Equations - As in the derivation of the Siacci equations (Section V), $\vec{P}$ is defined as a vector along the initial velocity vector of the projectile to a point (almost) directly above the shell, and $\vec{Q}$ is defined as a vector pointing vertically down from the tip of $\vec{P}$ (almost) to the projectile. The distance from the tip of $\vec{Q}$ to the projectile is the swerve $\vec{S}$. The swerve is defined to be the displacement of the shell from $\vec{P} + \vec{Q}$; i.e., it is the displacement of the shell from its straight-line path due to forces other than gravity (or due to aerodynamic forces but not to gravity). The Siacci equations ignore the swerve, but it is accounted for later by the windage jump. The procedure here is similar; differential equations for the swerve will be obtained from the differential equations of motion by subtracting out the Siacci equations for P and Q.

By definition, the initial velocity vector $\vec{u}_o$ is taken in the ξ, η plane so that $\alpha = \alpha_o = 0$. Since $\vec{P}$ lies along $\vec{u}_o$, the elevation angle of $\vec{P}$ above the horizontal (and in the ξ, η plane) is θ_o (see Fig. 12). Siacci-type equations for P and Q, and for swerve components S_ξ, S_η, and S_ζ are derived as follows. In terms of P, Q, S_ξ, S_η, and S_ζ, the coordinates ξ, η, and ζ are

$$\xi = P \cos \theta_o + S_\xi \tag{229}$$

$$\eta = P \sin \theta_o - Q + S_\eta \tag{230}$$

$$\zeta = S_\zeta \tag{231}$$

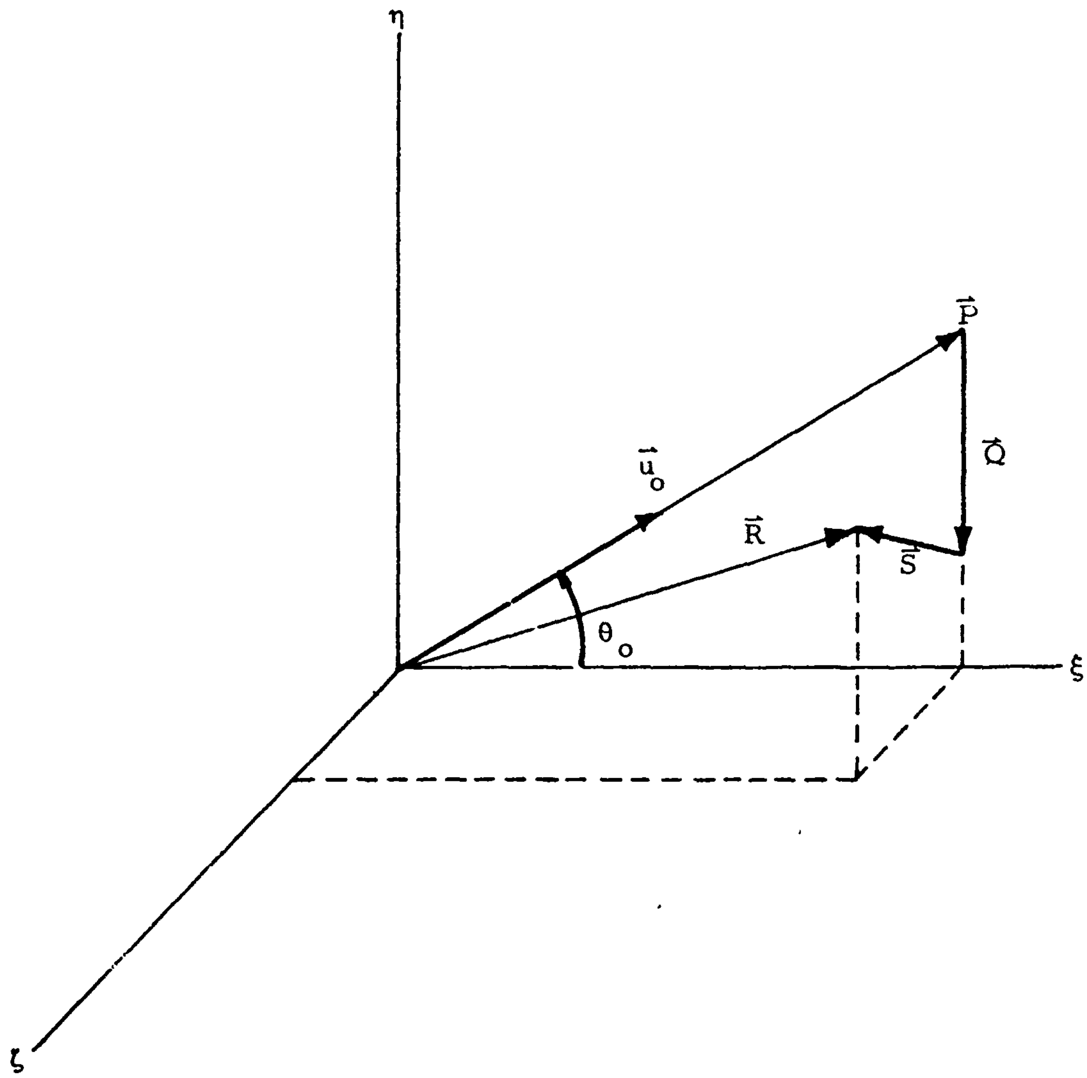


Figure 12

Coordinate System ξ, η, ζ Showing Projectile Range $\vec{R}$ in Terms of $\vec{P}$, $\vec{Q}$, and Swerve $\vec{S}$

Thus, the force Eqs. (223) through (228) become

$$m(\ddot{P}\cos\theta_o + \ddot{S}_\xi) = -\rho d^2 u K_D(\dot{P}\cos\theta_o + \dot{S}_\xi)$$

$$-\rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \sin\theta \cos a$$

$$+ \rho d^2 u^2 \left[K_L \sin\phi - \nu K_F \cos\phi\right] \sin\delta \sin a \tag{232}$$

$$m(\ddot{P}\sin\theta_o - \ddot{Q} + \ddot{S}_\eta) = -\rho d^2 u K_D(\dot{P}\sin\theta_o - \dot{Q} + \dot{S}_\eta)$$

$$+ \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \cos\theta - mg \tag{233}$$

$$m\ddot{S}_\zeta = -\rho d^2 u K_D \dot{S}_\zeta$$

$$+ \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \sin\theta \sin a$$

$$+ \rho d^2 u^2 \left[K_L \sin\phi - \nu K_F \cos\phi\right] \sin\delta \cos a \tag{234}$$

By definition, the Siacci equations for P and Q are

$$m\ddot{P} = -\rho d^2 u K_D \dot{P} \tag{235}$$

$$m\ddot{Q} = -\rho d^2 u K_D \dot{Q} + mg \tag{236}$$

with

$$u = (\dot{P}^2 - 2\dot{P}\dot{Q}\sin\theta_o + \dot{Q}^2)^{\frac{1}{2}} \tag{237}$$

They are obtained from the equations above with the swerve set to zero and with all aerodynamic coefficients except K_D set to zero. Upon substitution of Eqs. (235) and (236) into Eqs. (232) and (233), terms containing $\dot{P}$ and $\dot{Q}$ cancel and the equations for the swerve components S_ξ and S_η remain. The following are obtained:

$$m\ddot{S}_\xi = -\rho d^2 u K_D \dot{S}_\xi$$

$$-\rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \sin\theta \cos\alpha$$

$$+\rho d^2 u^2 \left[K_L \sin\phi - \nu K_F \cos\phi\right] \sin\delta \sin\alpha \tag{238}$$

$$m\ddot{S}_\eta = -\rho d^2 u K_D \dot{S}_\eta + \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \cos\theta \tag{239}$$

The equation for $\ddot{S}_\zeta$ is Eq. (234).

The equations derived up to this point would be exact if $\dot{S}_\xi$, $\dot{S}_\eta$, and $\dot{S}_\zeta$ were included in Eq. (237). Little error results, however, if these terms are ignored. For Eqs. (238) and (239) to be used as written, it would be necessary to integrate the equations for $\dot{\alpha}$ and $\dot{\theta}$. But since θ and α do not change much from their original values of $\theta = \theta_o$ and $\alpha = 0^o$ for those trajectories which are of interest, θ will be replaced by θ_o, cos α will be replaced by 1 and sin α by 0. The differential equations for the swerve become

$$m\ddot{S}_\xi = -\rho d^2 u K_D \dot{S}_\xi - \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \sin\theta_o \tag{240}$$

$$m\ddot{S}_\eta = -\rho d^2 u K_D \dot{S}_\eta + \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \cos\theta_o \tag{241}$$

$$m\ddot{S}_\zeta = -\rho d^2 u K_D \dot{S}_\zeta + \rho d^2 u^2 \left[K_L \sin\phi - \nu K_F \cos\phi\right] \sin\delta \tag{242}$$

Equations (240) and (241) can be combined. Take

$$S_\perp = -S_\xi \sin\theta_o + S_\eta \cos\theta_o \tag{243}$$

$$S_\parallel = S_\xi \cos\theta_o + S_\eta \sin\theta_o \tag{244}$$

where $S_\perp$ and $S_\parallel$ are in the ξ, η plane and are respectively perpendicular and parallel to $\vec{P}$; $S_\perp$ is positive above $\vec{P}$, and it will be shown that $S_\parallel$ is identically zero. The equations for $\ddot{S}_\perp$ and $\ddot{S}_\parallel$ as derived

from Eqs. (240) and (241) are

$$m\ddot{S}_{\perp} = -\rho d^2 u K_D \dot{S}_{\perp} + \rho d^2 u^2 \left[K_L \cos\phi + \nu K_F \sin\phi\right] \sin\delta \tag{245}$$

$$m\ddot{S}_{\parallel} = -\rho d^2 u K_D \dot{S}_{\parallel} \tag{246}$$

Since at t = 0, $\dot{S}_{\perp} = \dot{S}_{\parallel} = 0$, it is evident that $S_{\parallel}$ is identically zero for all t. Values of S_ξ and S_η, then, are

$$S_\xi = -S_{\perp} \sin\theta_o \tag{247}$$

$$S_\eta = S_{\perp} \cos\theta_o \tag{248}$$

and it is observed that the three differential equations for swerve have been reduced to two, Eqs. (242) and (245).

If Eqs. (242) and (245) are used in their present form to compute firing tables, the initial precession angle ϕ_o will have to be one of the table entry parameters; if one more coordinate transformation is made, however, this can be avoided and the volume of the tables will be greatly reduced. Components of swerve in and perpendicular to the plane of initial yaw are used. See Fig. 13 in which $\vec{P}$ is directed into the page. The new swerve components are

$$S_2 = S_{\perp} \cos\phi_o + S_\zeta \sin\phi_o \tag{249}$$

$$S_3 = -S_{\perp} \sin\phi_o + S_\zeta \cos\phi_o \tag{250}$$

with S_2 in the plane of yaw and S_3 normal to it. The differential equations for S_2 and S_3 are

$$m\ddot{S}_2 = -\rho d^2 u K_D \dot{S}_2 + \rho d^2 u^2 \left[K_L \cos\phi' + \nu K_F \sin\phi'\right] \sin\delta \tag{251}$$

$$m\ddot{S}_3 = -\rho d^2 u K_D \dot{S}_3 + \rho d^2 u^2 \left[K_L \sin\phi' - \nu K_F \cos\phi'\right] \sin\delta \tag{252}$$

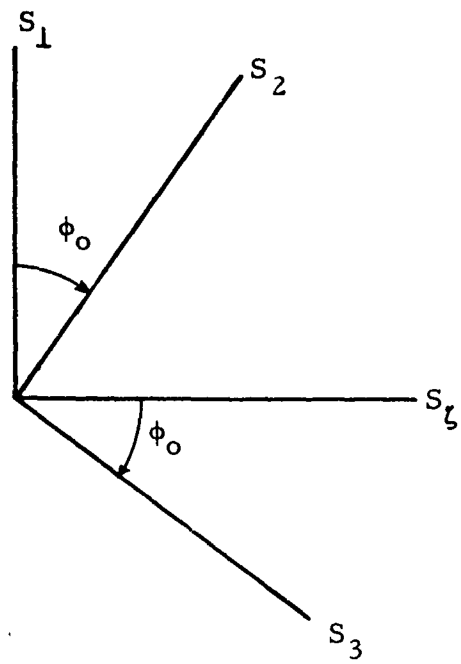


Figure 13
Relation of S_2 and S_3 with Respect to S_1 and S_ζ

where

$$\phi' = \phi - \phi_o$$

and $\phi' = 0$ at $t = 0$. Given ϕ_o, θ_o, and values of S_2 and S_3, values of S_ξ, S_η, and S_ζ can be found from

$$S_\xi = -(S_2 \cos\phi_o - S_3 \sin\phi_o) \sin\theta_o \tag{253}$$

$$S_\eta = (S_2 \cos\phi_o - S_3 \sin\phi_o) \cos\theta_o \tag{254}$$

$$S_\zeta = S_2 \sin\phi_o + S_3 \cos\phi_o \tag{255}$$

If angular windage jumps (in milliradians) are desired, they are

$$J_2 = 10^3 S_2/P \tag{256}$$

$$J_3 = 10^3 S_3/P \tag{257}$$

In Ref. 59, S_2 and S_3 were denoted by the symbols S_x and S_y, respectively. The notation used here is preferable, since S_2 and S_3 are in the 2 and 3 directions, respectively, of Fig. 11 when viewed at the time of firing. Calculations show that J_2 and J_3 approach more or less constant values as the projectile moves down its trajectory. Accordingly, windage jump can be accounted for by calculating the point-mass trajectory defined by Eqs. (235), (236), and (237), and by moving the resultant hit-point vector $\vec{P} + \vec{Q}$ through angles J_2 and J_3, in and perpendicular, respectively, to the initial plane of yaw, as seen from the point of fire. The windage jump correction amounts to a slight change in the direction of $\vec{P}$ or $\vec{u}$.

The set of equations derived thus far is complete and can be used to compute firing tables. If the spin is sufficiently large, however, the equations of angular motion can be simplified and the computation time can be reduced considerably. The following subsection contains the simplified equations and the justification for simplification.

4.2 The Approximate Equations of Angular Motion - The criterion for simplifying the equations of angular motion is that the spin must be large enough for the motion of the shell to be essentially gyroscopic. In this case, the angular momentum vector $\vec{H}$ will be nearly parallel to $\vec{1}_A$, and the component of $\vec{H}$ perpendicular to $\vec{1}_A$ will be small.

If $\vec{s}$ is a unit vector in the direction of $\vec{H}$,

$$\vec{H} = H\vec{s} \tag{258}$$

and Newton's second law for angular motion becomes

$$\vec{G} = \dot{\vec{H}} = \dot{H}\vec{s} + H\dot{\vec{s}} \tag{259}$$

and so

$$\vec{s} \times \vec{G} = H\vec{s} \times \dot{\vec{s}} = H\vec{\omega}_H \tag{260}$$

where

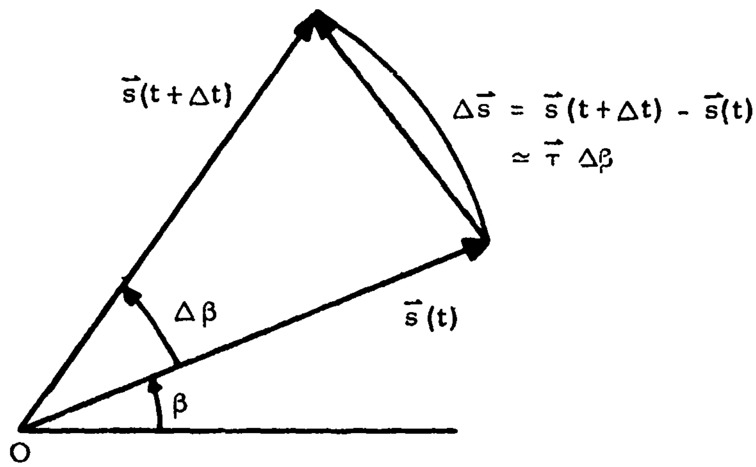
$$\vec{\omega}_H = \vec{s} \times \dot{\vec{s}} \tag{261}$$

is the angular velocity of $\vec{H}$, as can be seen from Fig. 14. Under the assumption that $A\omega_A >> B\omega_B$ (see Fig. 15), it follows that $\vec{H}$ is nearly parallel to $\vec{1}_A$, or that

$$\vec{s} \simeq \vec{1}_A \tag{262}$$

and that the angular velocity of $\vec{H}$, that is $\vec{\omega}_H$, is approximately equal to the transverse angular velocity of the projectile, $\vec{\omega}_T$;

$$\vec{\omega}_H \simeq \vec{\omega}_T = \vec{1}_B\omega_B + \vec{1}_3\omega_3 \tag{263}$$



$$\dot{\vec{s}} = \lim_{\Delta t \to 0} \frac{\vec{s}(t+\Delta t) - \vec{s}(t)}{\Delta t} = \lim_{\Delta t \to 0} \vec{\tau}\frac{\Delta\beta}{\Delta t} = \vec{\tau}\,\dot{\beta}$$

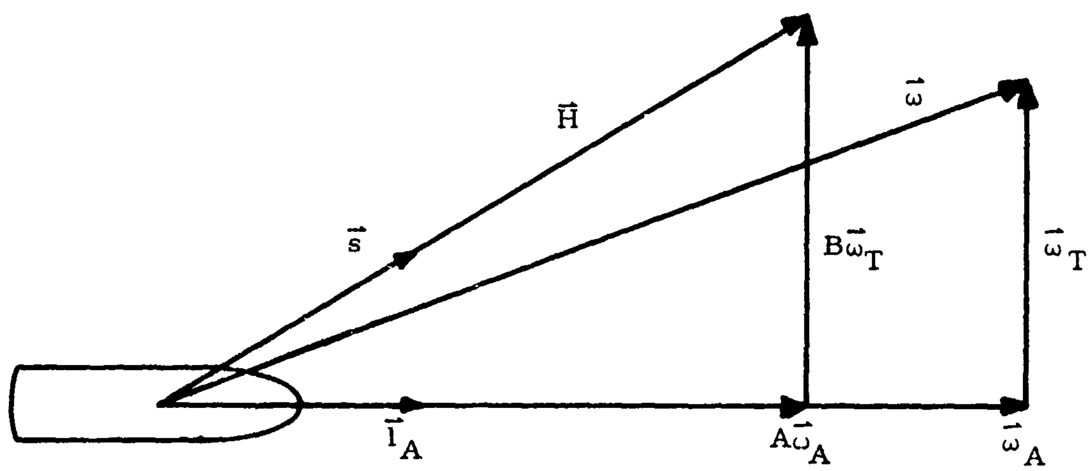
$$\vec{\omega}_H = \vec{s} \times \dot{\vec{s}} = \vec{s} \times \vec{\tau}\,\dot{\beta} = \vec{\nu}\,\dot{\beta}$$

$\vec{s}$ is a unit vector along $\vec{H}$

$\vec{\tau}$ is a unit vector along $\Delta\vec{s}$

$\vec{\nu}$ is a unit vector pointing out of the page

Figure 14
Proof that $\vec{s} \times \dot{\vec{s}} = \vec{\omega}_H$



$$\vec{\omega} = \vec{\omega}_A + \vec{\omega}_T$$

$$\vec{H} = A\vec{\omega}_A + B\vec{\omega}_T$$

Figure 15
The Relation Between $\vec{H}$ and $\vec{\omega}$

Then H $\approx A\omega_A$ and Eq. (260) becomes

$$A\omega_A\,(\vec{1}_B\omega_B + \vec{1}_3\omega_3) \simeq \vec{1}_A \times \vec{G} \tag{264}$$

It follows from Eqs. (203) and (204) that

$$\omega_3 A\omega_A \simeq \rho d^4 u\omega_A K_T \sin\delta - \rho d^4 u K_H \omega_B \tag{265}$$

$$-\omega_B A\omega_A \simeq \rho d^3 u^2 K_M \sin\delta - \rho d^4 u K_H \omega_3 \tag{266}$$

This approximation is equivalent to deleting terms containing B in Eqs. (213) and (214). The advantage is the elimination of terms containing $\dot{\omega}_B$ and $\dot{\omega}_3$. This negates the need for integrating to obtain ω_B and ω_3 and also smoothes out the nutational motion as is explained by Reed (Ref. 60). The smoothing of the nutational motion alleviates the requirement for a small step size (Δt) in numerical calculations.

The approximate equations for $\dot{\phi}$ and $\dot{\delta}$ are obtained by solving Eqs. (265) and (266) simultaneously for ω_B and ω_3.

$$\omega_B = \frac{-A\omega_A \rho d^3 u^2 K_M \sin\delta + \rho^2 d^8 u^2 \omega_A K_H K_T \sin\delta}{(A\omega_A)^2 + (\rho d^4 u K_H)^2} \tag{267}$$

$$\omega_3 = \frac{A\rho d^4 u\omega_A^2 K_T \sin\delta + \rho^2 d^7 u^3 K_H K_M \sin\delta}{(A\omega_A)^2 + (\rho d^4 u K_H)^2} \tag{268}$$

For high spin, $(\rho d^4 u K_H)^2$ is usually much less than one percent of $(A\omega_A)^2$; hence, $(\rho d^4 u K_H)^2$ is dropped from the denominator and these equations become

$$\omega_B = -\frac{\rho d^3 u^2}{A\omega_A}\left[K_M - \frac{\rho d^5}{A} K_H K_T\right]\sin\delta \tag{269}$$

$$\omega_3 = \frac{\rho d^4 u}{A}\left[K_T + \frac{\rho d^5}{A\nu^2} K_H K_M\right]\sin\delta \tag{270}$$

The equation for $\dot{\delta}$ can now be derived from Eqs. (182), (216), and (270). It becomes

$$\dot{\delta} = -\frac{\rho d^2 u}{m}\left[K_L - \frac{md^2}{A}\left\{K_T + \frac{\rho d^5}{A\nu^2} K_H K_M\right\}\right]\sin\delta + \frac{g}{u}\cos\theta\cos\phi \tag{271}$$

The gravity term must be dropped from this equation, since, for very small yaw, the aerodynamic term will be small compared to the gravity term and erroneous results will be obtained in numerical integration attempts. The term containing $K_H K_M$ is probably negligible. Calculations show this to be so for the 20-mm, M56 round, but a judgment will be needed for new rounds under development. To obtain the time derivative of cos δ, Eq. (271) is multiplied through by - sin δ. Then

$$\frac{d\cos\delta}{dt} = \frac{\rho d^2 u}{m}\left[K_L - \frac{md^2}{A}\left\{K_T + \frac{\rho d^5}{A\nu^2} K_H K_M\right\}\right]\sin^2\delta \tag{272}$$

The equation for $\dot{\phi}$ is derived from Eqs. (177), (178), (181), and (269) under the assumption that $\dot{\alpha}$ and $\dot{\theta}$ are negligible in comparison with $\dot{\phi}$. If $\dot{\alpha}$ and $\dot{\theta}$ are neglected, the equation for $\dot{\phi}$ is

$$\dot{\phi} = \frac{\rho d^3 u^2}{A\omega_A} K_M \tag{273}$$

The term containing K_H was dropped because it is negligible.

4.3 Summary of Equations - The approximate equations are summarized as follows for the convenience of the reader:

$$m\ddot{P} = -\rho d^2 u K_D \dot{P}$$

$$m\ddot{Q} = -\rho d^2 u K_D \dot{Q} + mg$$

$$m\ddot{S}_2 = -\rho d^2 u K_D \dot{S}_2 + \rho d^2 u^2 \left[K_L \cos\phi' + \nu K_F \sin\phi'\right] \sin\delta$$

$$m\ddot{S}_3 = -\rho d^2 u K_D \dot{S}_3 + \rho d^2 u^2 \left[K_L \sin\phi' - \nu K_F \cos\phi'\right] \sin\delta$$

$$\frac{d\cos\delta}{dt} = \frac{\rho d^2 u}{m} \left[K_L - \frac{md^2}{A}\left\{K_T + \frac{\rho d^5}{A\nu^2} K_H K_M\right\}\right] \sin^2\delta$$

$$\dot{\phi} = \frac{\rho d^3 u^2}{A\omega_A} K_M$$

$$u = \sqrt{\dot{P}^2 - 2\dot{P}\dot{Q}\sin\theta_o + \dot{Q}^2}$$

$$\nu = \frac{\omega_A d}{u}$$

$$\phi' = \phi - \phi_o$$

$$\xi = P\cos\theta_o + S_\xi$$

$$\eta = P\sin\theta_o + S_\eta - Q$$

$$\zeta = S_\zeta$$

$$S_\xi = -(S_2\cos\phi_o - S_3\sin\phi_o)\sin\theta_o$$

$$S_\eta = (S_2\cos\phi_o - S_3\sin\phi_o)\cos\theta_o$$

$$S_\zeta = S_2\sin\phi_o + S_3\cos\phi_o$$

SECTION V

THE SIACCI METHOD

1. General

The Siacci method (Refs. 1, 24, 27, and 61) is a means of obtaining approximate solutions to projectile trajectories by use of tables. It is applicable in many situations where high computational accuracy is not essential, such as in preliminary design and in new concept studies, and in particular, it is valid for relatively short trajectories where gravity drop is not appreciable and yaw is small. The calculational ease with which the method can be utilized, e.g., in hand computations, or in an airborne fire-control computer, makes the Siacci method invaluable in many modern applications despite its early origin. The original method was devised sometime around 1880 by F. Siacci of Italy. The treatment given here is that of Ref. 61 with minor changes.

With the development of the 20-mm, M56 round in the late 1950's, the trend in ballistics calculations was away from use of the Siacci method and toward more sophisticated calculations since modern computing equipment was becoming available. Sophisticated methods are now well suited to ground-based ballistics investigations, but for airborne fire-control calculations, onboard computers are still somewhat limited in capacity. The Siacci method is still a candidate for airborne fire-control calculations.

A description of the basic Siacci method follows and a derivation is given in Subsection 2. In its basic form, the trajectory as given by the Siacci solution is as follows:

$$t = \frac{C}{\sigma a_o} \left[T(u/a_o) - T(u_o/a_o) \right] \tag{274}$$

$$P = \frac{C}{\sigma} \left[S(u/a_o) - S(u_o/a_o) \right] \quad (275)$$

and

$$Q = \left(\frac{C}{\sigma a_o}\right)^2 \left[A(u/a_o) - A(u_o/a_o) - I(u_o/a_o) \frac{\sigma}{C} P \right] \quad (276)$$

in which S, T, I, and A are tabulated functions of $U = u/a_o$; t is the time of flight, P is the "pseudorange" along the initial velocity vector $\vec{u}_o$, and Q is the gravity drop (Fig. 16). The parameter σ is the relative air density at the firing altitude, a_o is the ratio of the speed of sound at the firing altitude to that at sea level, and C is the ballistic coefficient given in terms of the projectile mass m, in pounds, and diameter d, in feet, as shown below:

$$C = \frac{m}{144d^2} \left(lb/in.^2\right)$$

The parameter u is defined as follows

$$u = \frac{dP}{dt}$$

and is the independent variable.

The tabulated functions S, T, I, and A are calculated by numerical integration of the following equations:

$$\frac{dS}{dU} = - \frac{1}{G(U)} \quad (277)$$

$$\frac{dT}{dU} = - \frac{1}{UG(U)} \quad (278)$$

$$\frac{dI}{dU} = - \frac{g}{U^2 G(U)} \quad (279)$$

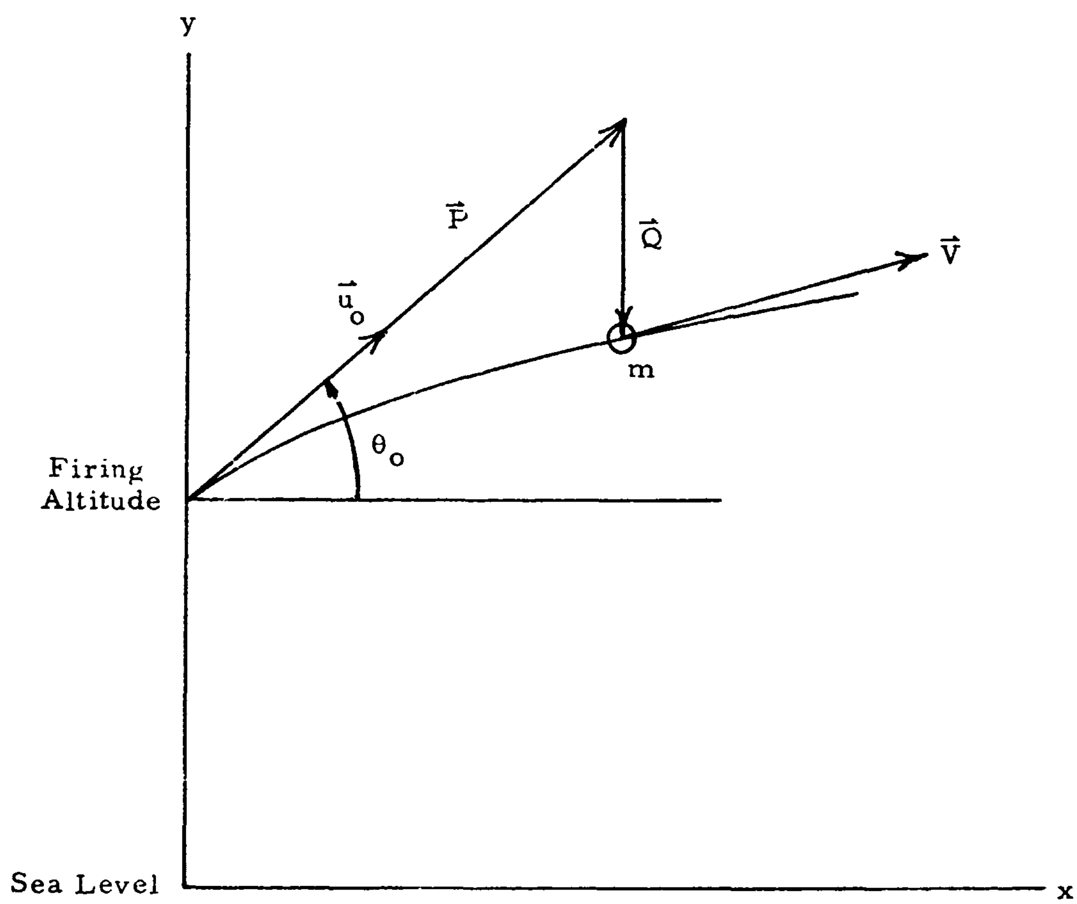


Figure 16
Siacci Coordinates

$$\frac{dA}{dU} = -\frac{I(U)}{G(U)} \tag{280}$$

in which

$$g = 32.174 \text{ ft/sec}^2$$

is the acceleration due to gravity, and

$$G(U) = \frac{UK_D\left(U/V_{so}\right)}{1883} \tag{281}$$

K_D is the zero-yaw drag coefficient, and V_{so} = 1,116.45 ft/sec is the speed of sound at sea level. The numerical integration is carried out backwards (i.e., the integration proceeds from larger to smaller U) from some arbitrarily chosen value of U, e.g., V, such that $V > u/a_o$ for all u of interest. Initial conditions are S(V) = 0, T(V) = 0, A(V) = 0, and I(V) = 0. The solution given by Eqs. (274), 275), and (276) is independent of the choice of V. The Siacci functions bear the following names:

S	Space function
T	Time function
I	Inclination function
A	Altitude function

At the date of writing, tables have been prepared for the 7.62-mm NATO round, the 20-mm, M56 round, and the 40-mm, Mk 2 round, (Refs. 62, 63, and 64).

For computer applications (airborne or ground based) the Siacci functions can be curve fitted.

The Siacci method in its basic form is useful in most situations where point-mass (particle) trajectories are applicable, that is, situations where the projectile angle of attack (yaw) is small and all aerodynamic forces except the drag are negligible. This situation occurs for forward fire from fixed-gun fighters and sometimes in tail defense of bombers. For example, the Siacci method is applicable to fire-control problems for the F-111 and for use with tracer-line-type gun

sights, although it may be just as convenient to perform onboard particle trajectory computations in these instances.

For fire from flexible guns and from fixed gun fighters in high angle-of-attack situations, the angle of attack of the projectile can be large and other aerodynamic forces besides the drag become important. In this instance the Siacci method in its basic form becomes inadequate. Onboard trajectory computations are no longer reasonable, however, when six-degree-of-freedom computations are necessary. Techniques for improving the accuracy of the Siacci method for the large-yaw situation are given in Subsection 3.

2. The Basic Siacci Method

The basic Siacci trajectory is an approximation of a point-mass trajectory. Corrections can be added to improve its accuracy as will be discussed later in Subsection 3. The equations of motion of a mass point, as can be derived from Fig. 16, are

$$m\ddot{x} = -E\dot{x} \tag{282}$$

$$m\ddot{y} = -E\dot{y} - mg \tag{283}$$

where

$$E = \rho d^2 V K_D(V/V_s) \tag{284}$$

V is the projectile velocity, d is the projectile diameter, m is the projectile mass, g is the acceleration due to gravity, K_D is the drag coefficient, and ρ and V_s are respectively, the air density and the speed of sound at the altitude of the firing point. Both ρ and V_s are assumed constant over the trajectory. This is not a bad assumption for the air-to-air case and for close air support.

Coordinates x and y are to be replaced by Siacci coordinates P and Q shown in Fig. 16. Substitution of

$$x = P\cos\theta_o \tag{285}$$

$$y = P \sin \theta_o - Q \tag{286}$$

into the differential equations yields

$$m\ddot{P} = -E\dot{P} \tag{287}$$

$$m\ddot{Q} = -E\dot{Q} + mg \tag{288}$$

where θ_o is the angle between $\vec{P}$ and the x axis and is a constant.

An equation for $\dot{Q}/\dot{P}$ is required. By means of the relation

$$\frac{d}{dt}\left(\frac{\dot{Q}}{\dot{P}}\right) = \frac{\dot{P}\ddot{Q} - \dot{Q}\ddot{P}}{\dot{P}^2}$$

and the differential equations above, it can be shown that

$$\frac{d}{dt}\left(\frac{\dot{Q}}{\dot{P}}\right) = \frac{g}{\dot{P}}$$

With the substitution $u = \dot{P}$, the equations to be solved are

$$m\dot{u} = -\rho d^2 uVK_D(V/V_s) \tag{289}$$

$$\dot{P} = u \tag{290}$$

and

$$\frac{d}{dt}\left(\frac{\dot{Q}}{u}\right) = \frac{g}{u} \tag{291}$$

in which E has been replaced by $\rho d^2 VK_D$, and V is given by

$$V = \sqrt{\dot{P}^2 - 2\dot{P}\dot{Q} \sin \theta_o + \dot{Q}^2} \tag{292}$$

For short, flat trajectories, $\dot{Q} << \dot{P}$, and,to a good approximation,

$$V \simeq \dot{P} = u \tag{293}$$

This is the Siacci approximation which enables the trajectory computation to be reduced to tabular form since, from Eq. (289),

$$dt = -\frac{m\,du}{\rho\, d^2 u^2 K_D(u/V_s)} \tag{294}$$

can be integrated directly.

Before solution, it is desirable to make the following substitutions:

$$t = \frac{C}{\sigma a_o}\left[T - T_o\right] \tag{295}$$

$$P = \frac{C}{\sigma}\left[S - S_o\right] \tag{296}$$

$$Q = \left(\frac{C}{\sigma a_o}\right)^2 q \tag{297}$$

$$u = a_o U \tag{298}$$

where C, σ, a_o, T_o, and S_o are constants and T, S, q, and U are new variables which replace t, P, Q, and u, respectively. C is the ballistic coefficient given by

$$C = \frac{m}{(12d)^2} \tag{299}$$

Units of C are lb/in.^{2} (for consistency with past usage) when m is in pounds and d is in feet; σ is the ratio of the air density at altitude, ρ, to that at sea level, ρ_o,

$$\sigma = \frac{\rho}{\rho_o} \tag{300}$$

where $\rho_o = 0.076474$ lb/ft^3. The parameter a_o is the ratio of the speed of sound V_s, at altitude, to that at sea level, $V_{so} = 1,116.45$ ft/sec,

$$a_o = \frac{V_s}{V_{so}} \tag{301}$$

With these substitutions, the equations of motion become

$$\frac{dU}{dT} = - UG(U) \tag{302}$$

$$\frac{dS}{dT} = U \tag{303}$$

and

$$\frac{d}{dT}\left(\frac{1}{U}\frac{dq}{dT}\right) = \frac{g}{U} \tag{304}$$

where

$$G(U) = \frac{UK_D(U/V_{so})}{1883} \tag{305}$$

The reason for the substitution is now clear; these equations are independent of V_s and ρ and hence, the Siacci tables are independent of altitude.

From the first equation

$$dT = - \frac{dU}{UG(U)}$$

and hence

$$T - T_o = \int_U^{U_o} \frac{dU}{UG(U)}$$

where $U_o = u_o/a_o$, and u_o is the initial projectile velocity. If T is defined as

$$T(U) = \int_U^V \frac{dU}{UG(U)} \tag{306}$$

where V, for all applications of interest, is any convenient number such that $V > U_o$, it follows that

$$T - T_o = T(U) - T(U_o)$$

$$= \int_U^V \frac{dU}{UG(U)} - \int_{U_o}^V \frac{dU}{UG(U)}$$

That $T - T_o$ is independent of V is obvious.

The equation for S may be solved in a similar manner by eliminating dT.

$$dS = U\,dT = -\frac{dU}{G(U)}$$

$$S - S_o = \int_U^{U_o} \frac{dU}{G(U)}$$

S is defined as

$$S(U) = \int_U^V \frac{dU}{G(U)} \tag{307}$$

therefore

$$S - S_o = S(U) - S(U_o)$$

For the third expression, we have

$$d\left(\frac{1}{U}\,\frac{dq}{dT}\right) = \frac{g}{U}\,dT = -\frac{g\,dU}{U^2 G(U)}$$

so

$$\frac{1}{U}\frac{dq}{dT} = \int_{U}^{U_o} \frac{gdU}{U^2 G(U)}$$

since $dq/dT = 0$ at $T = T_o$. If a function $I(U)$ is defined as

$$I(U) = \int_{U}^{V} \frac{gdU}{U^2 G(U)} \tag{308}$$

we have

$$\frac{1}{U}\frac{dq}{dT} = I(U) - I(U_o)$$

Thus

$$\begin{aligned} dq &= UI(U)dT - UI(U_o)dT \\ &= -\frac{I(U)\,dU}{G(U)} + \frac{I(U_o)\,dU}{G(U)} \\ &= -\frac{I(U)\,dU}{G(U)} - I(U_o)\,dS \end{aligned}$$

and it follows that

$$q = \int_{U}^{U_o} \frac{I(U)\,dU}{G(U)} - I(U_o)\,(S - S_o)$$

If a function

$$A(U) = \int_U^V \frac{I(U)\,dU}{G(U)} \tag{309}$$

is defined, then

$$q = A(U) - A(U_o) - I(U_o)\left\{S(U) - S(U_o)\right\}$$

It follows that the relations for t, P, and Q are

$$t = \frac{C}{\sigma a_o}\left[T(u/a_o) - T(u_o/a_o)\right] \tag{310}$$

$$P = \frac{C}{\sigma}\left[S(u/a_o) - S(u_o/a_o)\right] \tag{311}$$

$$Q = \left(\frac{C}{\sigma a_c}\right)^2 \left[A(u/a_o) - A(u_o/a_o) - I(u_o/a_o)\,\frac{\sigma}{C}\,P\right] \tag{312}$$

where T, S, I, and A are given by Eqs. (306) through (309), respectively. In application, the functions S, T, A, and I are tabulated once and for all by numerical integration. Given initial and final bullet velocities u_o and u, respectively, t, P, and Q can be calculated by use of the tables.

3. Corrections to the Siacci Method

This section contains techniques for improving the accuracy of the Siacci method when some of the basic assumptions upon which the method is founded are violated. Corrections are developed for a variable atmosphere, yaw drag. and windage jump. This section is not complete in the sense that the methods are fully developed. Rather it is meant to serve as an indication of how correction techniques are derived. Indeed, techniques depend somewhat upon individual rounds and are best investigated

in conjunction with other calculational methods such as six-degree-of-freedom calculations. Correction terms needed for one type of projectile may not be needed for another. Also, correction terms may be derived by a number of different methods. Thus, different formulae for the correction terms derived herein can be found in Refs. 65 and 66. A comparison of correction techniques is given in Ref. 67 for some obsolete rounds.

3.1 A Variable Air-Density Correction - The Siacci theory in its basic form is founded upon the approximation that the air density is constant along the projectile trajectory. To compensate for this approximation, correction formulae may be derived. The following comments are pertinent, however.

In problems involving air-to-air combat, the target and attack aircraft are usually at about the same altitude and the air-density correction is unnecessary. In low-altitude air-to-ground problems, experience has shown that the calculated hit point is not much affected by the constant air-density approximation, whereas the calculated time of flight is affected. But, in air-to-ground situations, the time of flight is usually not needed to any great accuracy provided the target is not moving rapidly and the wind is not strong. It is concluded that in most cases an air-density correction is not needed, but nevertheless correction formulae will be developed for situations where they may be needed. The derivation follows.

It will be assumed that the expression

$$\rho = \rho_o \sigma e^{-h\Delta y} \tag{313}$$

adequately represents the air-density variation at points between the gun and the target, where ρ_o is sea level air density, σ is the relative air density at the gun, h is a constant ($h \approx 3.158 \times 10^{-5}\ \mathrm{ft}^{-1}$), and Δy is the altitude variation along the trajectory. Evidently, from Fig. 16,

$$\Delta y = P \sin\theta_o - Q \simeq P \sin\theta_o$$

Now consider the unyawed equations of motion (289), (290), and (291). Eq. (289) becomes

$$m\dot{u} = -\rho_o \sigma e^{-hP\sin\theta_o} d^2 u^2 K_D(u/V_s)$$

under the Siacci approximation u = V. It follows that

$$-\frac{C}{\sigma a_o} \frac{dU}{UG(U)} = e^{-hP\sin\theta_o} dt$$

and that

$$-\frac{C}{\sigma} \frac{dU}{G(U)} = e^{-hP\sin\theta_o} dP$$

Integration of the second expression leads to

$$\frac{C}{\sigma}\left[S(U) - S(U_o)\right] = \frac{1 - e^{-hP\sin\theta_o}}{h\sin\theta_o} \tag{314}$$

Solving for $e^{-hP\sin\theta_o}$ yields

$$e^{-hP\sin\theta_o} = 1 - \frac{C}{\sigma} h\sin\theta_o\left[S(U) - S(U_o)\right]$$

This expression is of the form

$$e^{-x} = 1 - y$$

Hence

$$e^{x} = \frac{1}{1-y} = 1 + y + y^2 + y^3 + \ldots$$

provided $|y| < 1$. For an air-to-ground attack from an altitude of, say, $P \sin\theta_o = 5,000$ ft

$$x = hP \sin\theta$$

$$= (3.158 \times 10^{-5})(5,000) = 0.16$$

and

$$e^{-x} = 0.852$$

It follows that

$$y = 0.148$$

and

$$y^2 = 0.0219$$

Then, to within about 2 percent

$$e^x = 1 + y$$

or

$$e^{hP \sin\theta_o} = 1 + \frac{C}{\sigma} h \sin\theta_o \left[S(U) - S(U_o)\right]$$

The expression for dt becomes

$$dt = -e^{hP \sin\theta_o} \frac{C}{\sigma a_o} \frac{dU}{UG(U)}$$

$$= -\left\{1 + \frac{C}{\sigma} h \sin\theta_o \left[S(U) - S(U_o)\right]\right\} \frac{C}{\sigma a_o} \frac{dU}{UG(U)}$$

and hence

$$t = \frac{C}{\sigma a_o}\left[T(U) - T(U_o)\right] + \left(\frac{C}{\sigma}\right)^2 \frac{h \sin\theta_o}{a_o}\left\{\left[H(U) - H(U_o)\right] - S(U_o)\left[T(U) - T(U_o)\right]\right\} \quad (315)$$

where

$$H(U) = \int_U^V \frac{S(U)\, dU}{UG(U)} \quad (316)$$

H(U) may be tabulated once and for all along with S, T, I, and A.

The numerical example above applies to maximum P and hence,to minimum U. As U runs from U_o to U, y varies from zero to 0.148. This implies that the integration is probably much more accurate than 2 percent. If accuracy is found insufficient, e.g., for higher altitude air-to-ground fire, more terms can be taken in the expansion for $(1 - y)^{-1}$. The analysis proceeds in a straightforward manner.

The correction term for Q may be derived as follows:

$$d\left(\frac{\dot{Q}}{u}\right) = \frac{g}{u}\, dt$$

$$= -\frac{g}{a_o U}\left\{1 + \frac{C}{\sigma} h \sin\theta_o\left[S(U) - S(U_o)\right]\right\}\frac{C}{\sigma a_o}\frac{dU}{UG(U)}$$

Then

$$\frac{\dot{Q}}{u} = \frac{C}{\sigma a_o^2}\left[I(U) - I(U_o)\right] + \left(\frac{C}{\sigma a_o}\right)^2 h \sin\theta_o\left[\left\{S(U) - S(U_o)\right\} I(U) - \left\{A(U) - A(U_o)\right\}\right] \quad (317)$$

If, for short, $\dot{Q}/u = F$, then

$$dQ = Fu\, dt$$

$$= -F\left\{1 + \frac{C}{\sigma} h \sin\theta_o \left[S(U) - S(U_o)\right]\right\} \frac{C}{\sigma} \frac{dU}{G(U)}$$

If the second-order term in $h \sin \theta_o$ is discarded, since it is obviously small, then

$$dQ = -\left(\frac{C}{\sigma a_o}\right)^2 \left[I(U) - I(U_o)\right] \frac{dU}{G(U)}$$

$$-\left(\frac{C}{\sigma a_o}\right)^3 a_o h \sin\theta_o \left[\left\{S(U) - S(U_o)\right\} I(U) - \left\{A(U) - A(U_o)\right\}\right] \frac{dU}{G(U)}$$

$$-\left(\frac{C}{\sigma a_o}\right)^3 a_o h \sin\theta_o \left[I(U) - I(U_o)\right] \left[S(U) - S(U_o)\right] \frac{dU}{G(U)}$$

and finally,

$$Q = \left(\frac{C}{\sigma a_o}\right)^2 \left[\left\{A(U) - A(U_o)\right\} - I(U_o)\left\{S(U) - S(U_o)\right\}\right]$$

$$+ \left(\frac{C}{\sigma a_o}\right)^3 a_o h \sin\theta_o \left[\left\{W_1(U) - W_1(U_o)\right\}\right.$$

$$- 2S(U_o)\left\{A(U) - A(U_o)\right\} - I(U_o)\left\{W_2(U) - W_2(U_o)\right\}$$

$$\left. + \left\{A(U_o) + I(U_o) S(U_o)\right\}\left\{S(U) - S(U_o)\right\}\right] \tag{318}$$

where

$$W_1(U) = \int_U^V \frac{2S(U)\, I(U) - A(U)}{G(U)} dU \tag{319}$$

and

$$W_2(U) = \int_U^V \frac{S(U)}{G(U)}\, dU \tag{320}$$

The correction term is quite complicated, and hopefully, is negligible in most cases.

3.2 <u>A Yaw-Drag Correction</u> - In more exact trajectory computations, other aerodynamic forces in addition to the zero-yaw drag force are included, and the projectile is treated as a rigid body which executes angular motion about its center of mass. An important additional force is the yaw drag which arises from the angle of attack (yaw) of the projectile. The yaw angle δ is measured between the projectile body axis and the projectile velocity vector. The drag force for the yawed projectile can be written as

$$K_D(M, \delta) = K_{D_o}(M) \left[1 + K_{D_{\delta^2}}\, \delta^2\right] \tag{321}$$

where, for present purposes, $K_{D_{\delta^2}}$ is considered to be a constant and $M = V/V_s$ is the Mach number. This expression for K_D takes the place of K_D in the equations of motion, i.e., in Eqs. (282), (283), and (284), and also in Eqs. (287) and (288). If E retains its meaning, Eqs. (287) and (288) become

$$m\ddot{P} = -E \left[1 + K_{D_{\delta^2}}\, \delta^2\right] \dot{P}$$

$$m\ddot{Q} = -E \left[1 + K_{D_{\delta^2}}\, \delta^2\right] \dot{Q} + mg$$

If the derivation proceeds as before with the Siacci approximation $V = \dot{P}$, Eqs. (289), (290), and (291) show that the following relations will be obtained:

$$m\dot{u} = -\rho d^2 u^2 K_D(u/V_s)\left[1 + K_{D_{\delta^2}}\delta^2\right]$$

$$\dot{P} = u$$

$$\frac{d}{dt}\left(\frac{\dot{Q}}{u}\right) = \frac{g}{u}$$

Substitution of $a_o U$ for u, $\rho_o\sigma$ for ρ, $144d^2C$ for m, and use of Eq. (305) for K_D yields the following result for the first relation above

$$\left[1 + K_{D_{\delta^2}}\delta^2\right] dt = -\frac{C}{\sigma a_o}\frac{dU}{UG(U)}$$

It follows immediately that

$$t = \frac{C}{\sigma a_o}\left[T(u/a_o) - T(u_o/a_o)\right] - K_{D_{\delta^2}}\int_0^t \delta^2\, dt \qquad (322)$$

With

$$dt = \frac{dP}{u} = \frac{dP}{a_o U}$$

it follows that

$$\left[1 + K_{D_{\delta^2}}\delta^2\right] dP = -\frac{C}{\sigma}\frac{dU}{G(U)}$$

and

$$P = \frac{C}{\sigma}\left[S(u/a_o) - S(u_o/a_o)\right] - K_{D_{\delta^2}}\int_0^P \delta^2\, dP \qquad (323)$$

In order to derive a correction term for Q it is convenient to write the differential equation for Q in a different form as follows

$$\frac{d}{dt}\left(\frac{Q}{u}\right) = \frac{d}{dt}\left(\frac{\dot{Q}}{\dot{P}}\right) = \frac{d}{dt}\frac{dQ}{dP} = \frac{dP}{dt}\frac{d}{dP}\left(\frac{dQ}{dP}\right)$$

$$= u\,\frac{d^2Q}{dP^2} = \frac{g}{u}$$

or

$$\frac{d^2Q}{dP^2} = \frac{g}{u^2}$$

Let

$$D = \frac{dQ}{dP}$$

so that

$$\frac{dD}{dP} = \frac{g}{u^2}$$

Then

$$\left[1 + K_{D_{\delta^2}}\,\delta^2\right] dD = \frac{g}{u^2}\left[1 + K_{D_{\delta^2}}\,\delta^2\right] dP$$

$$= -\frac{g}{u^2}\,\frac{C}{\sigma}\,\frac{dU}{G(U)}$$

or

$$dD = -K_{D_{\delta^2}}\,\delta^2\,\frac{g}{u^2}\,dP - \frac{C}{\sigma a_o^2}\,\frac{g\,dU}{U^2G(U)}$$

and it follows that

$$D = -K_{D_{\delta^2}}\int_0^P \delta^2\,\frac{g}{u^2}\,dP + \frac{C}{\sigma a_o^2}\left[I(U) - I(U_o)\right] \qquad (324)$$

Also

$$\left[1 + K_{D_{\delta^2}}\right] dQ = D\left[1 + K_{D_{\delta^2}}\right] dP$$

$$= - K_{D_{\delta^2}} \left[1 + K_{D_{\delta^2}} \delta^2\right] \int_0^P \delta^2 \frac{g}{u^2} \, dP' \, dP$$

$$- \frac{C}{\sigma a_o^2} \left[I(U) - I(U_o)\right] \frac{C}{\sigma} \frac{dU}{G(U)}$$

It follows that

$$Q = - K_{D_{\delta^2}} \int_0^Q \delta^2 \, dQ - K_{D_{\delta^2}} \int_0^P \left[1 + K_{D_{\delta^2}} \delta^2\right] \int_0^{P'} \delta^2 \frac{g}{u^2} \, dP'' \, dP'$$

$$+ \left(\frac{C}{\sigma a_o}\right)^2 \left[\left\{A(U) - A(U_o)\right\} - I(U_o)\left\{S(U) - S(U_o)\right\}\right]$$

or

$$Q = - K_{D_{\delta^2}} \frac{C}{\sigma a_o^2} I(u_o/a_o) \int_0^P \delta^2 \, dP$$

$$- K_{D_{\delta^2}} \int_0^Q \delta^2 \, dQ - K_{D_{\delta^2}} \int_0^P \left[1 + K_{D_{\delta^2}} \delta^2\right] \int_0^{P'} \delta^2 \frac{g}{u^2} \, dP'' \, dP'$$

$$+ \left(\frac{C}{\sigma a_o}\right)^2 \left[A(u/a_o) - A(u_o/a_o) - I(u_o/a_o) \frac{\sigma}{C} P\right] \tag{325}$$

It is seen that equations for t, P, and Q can be written as before but with correction terms

$$\Delta t_1 = -K_{D_{\delta^2}} \int_0^t \delta^2 \, dt \tag{326}$$

$$\Delta P_1 = -K_{D_{\delta^2}} \int_0^P \delta^2 \, dP \tag{327}$$

$$\Delta Q_1 = -K_{D_{\delta^2}} \frac{C}{\sigma a_o^2} I(u_o/a_o) \int_0^P \delta^2 \, dP$$

$$- K_{D_{\delta^2}} \int_0^Q \delta^2 \, dQ - K_{D_{\delta^2}} \int_0^P \left[1 + K_{D_{\delta^2}} \delta^2\right] \int_0^{P'} \delta^2 \frac{g}{u^2} \, dP'' \, dP' \tag{328}$$

These equations can be evaluated only if δ is a known function. According to an approximate analysis by Sterne (Ref. 9), δ is given approximately by the relation

$$\overline{\delta^2} = \delta_o^2 \frac{s_o - 1/2}{s_o - 1} e^{-2\sigma c P} \tag{329}$$

where $\overline{\delta^2}$ is an average, squared yaw, s_o represents the static stability factor which is given by

$$s_o = \frac{A^2 N^2}{4B \, \rho d^3 u_o^2 K_M} \tag{330}$$

and

$$c = c' + \frac{c''}{s_o - 1} \tag{331}$$

where N is the axial spin,

$$c' = \frac{\rho_o d^2}{2m} \left[\frac{md^2}{B} K_H + K_L \right] \tag{332}$$

and

$$c'' = \frac{\rho_o d^2}{2m} K_D \tag{333}$$

When Sterne's analysis was derived, there was no convenient method of solving the general six-degree-of-freedom equations since there were no large-scale digital computers. Now, it may be convenient to solve them for yaw dependence and curve fit the solution. In any case, it is probably true that $\bar{\delta}$ is given approximately by

$$\bar{\delta} = K\delta_o e^{-\beta P} \tag{334}$$

where K and β are constants.

From the approximate theory, the equation for yaw, Eq. (272), may be written as

$$\dot{\bar{\delta}} = -\frac{\rho d^2 u}{m} \left[K_L - \frac{md^2}{A} K_T \right] \bar{\delta}$$

where the gravity term and the damping term K_H have been deleted and the approximation $\sin \bar{\delta} = \bar{\delta}$ has been used. If Eq. (334) is assumed for $\bar{\delta}$, then

$$\dot{\bar{\delta}} = -\beta \dot{P} K \delta_o e^{-\beta P} = -\beta u \bar{\delta}$$

and it follows that

$$\beta = \frac{\rho d^2}{m} \left[K_L - \frac{md^2}{A} K_T \right]$$

It is interesting to compare this result with that obtained from the Sterne theory, Eqs. (329) through (333), in which $\beta = \sigma c$. Curiously enough, it can be shown that the two are in reasonable numerical agreement, at least for the 20-mm, M56 round, despite the different emphasis on K_H in the two theories.

From examination of Eqs. (326), (327), and (328), it is evident that only ΔP_1 can be integrated directly:

$$\Delta P_1 = -K_{D_{\delta^2}} \delta_o^2 \frac{s_o - 1/2}{s_o - 1} \int_0^P e^{-2\sigma c P} dP = -\frac{k_o}{2\sigma c}\left[1 - e^{2\sigma c P}\right] \quad (335)$$

where

$$k_o = \frac{s_o - 1/2}{s_o - 1} K_{D_{\delta^2}} \delta_o^2 \quad (336)$$

The equation for $S(u/a_o)$ is often written as

$$S(u/a_o) = S(u_o/a_o) + \frac{\sigma}{C} P + \frac{k_o}{2cC} \quad (337)$$

in which the $e^{-2\sigma c P}$ term is neglected for a large enough P.

An expression for Δt_1 can be derived as follows: since $\dot{P} = u$, and with $k = 2\sigma c$,

$$\Delta t_1 = -K_{D_{\delta^2}} \delta_o^2 \frac{s_o - 1/2}{s_o - 1} \tau = -k_o \tau$$

where

$$a_o \tau \quad \int_0^P \frac{e^{-kP}}{U} dP$$

The expression for S(U) with the ΔP_1 correction is

$$S(U) = S(U_o) + \frac{\sigma}{C} P + \frac{k_o}{2cC}\left\{1 - e^{-2\sigma c P}\right\} \quad (338)$$

By use of this expression, a series in P can be derived for 1/U. It is

$$\frac{1}{U} = \frac{1}{U_o} + \frac{G(U_o)}{U_o^2}\left\{(1 + k_o)\frac{\sigma}{C} P\right\}$$

$$+ \frac{G(U_o)}{2U_o^2}\left\{\frac{2}{U_o} G(U_o) - G'(U_o) - \frac{2cCk_o}{(1+k_o)^2}\right\}\left\{(1+k_o)\frac{\sigma}{C} P\right\}^2$$

$$+ \cdots$$

where

$$G'(U_o) = \frac{dG(U_o)}{dU_o}$$

If only the first two terms of this series are used,

$$a_o\tau \approx \int_0^P \left\{\frac{1}{U_o} + \frac{G(U_o)}{U_o^2}(1 + k_o)\frac{\sigma}{C} P\right\} e^{-kP} dP$$

The function represented by the first two terms of the series for 1/U is a straight line with the correct magnitude and slope at P = 0. As P increases, it deviates from 1/U. But as P increases, the exponential term decreases, so the major contribution to the integral is obtained for P near zero where the approximation to 1/U is more accurate. It follows that

$$\Delta t_1 = -k_o\left[\frac{1 - e^{-kP}}{ku_o} + \frac{(1 + k_o) G(u_o/a_o)}{k^2 u_o^2}\frac{\sigma a_o}{C}\left\{1 - (1 + kP)e^{-kP}\right\}\right]$$

If this expression is not accurate enough, more terms can be retained. For large kP, it reduces to

$$\Delta t_1 = -k_o \left[\frac{1}{k u_o} + \frac{(1 + k_o) G(u_o/a_o)}{k^2 u_o^2} \frac{\sigma a_c}{C} \right] \tag{339}$$

An expression can be derived in a similar manner for ΔQ_1. By use of Eqs. (329) and (336), Eq. (328) becomes

$$\Delta Q_1 = - \frac{C}{\sigma a_o^2} I(u_o/a_o) k_o \int_0^P e^{-kP} dP$$

$$- k_o \int_0^P e^{-kP} \frac{dQ}{dP} dP - k_o \int_0^P \left[1 + k_o e^{-kP'} \right] \int_0^{P'} e^{-kP''} \frac{g}{u^2} dP'' dP'$$

Let Z_1, Z_2, and Z_3 represent, respectively, the three integrals. The first is

$$Z_1 = \int_0^P e^{-kP} dP = \frac{1 - e^{-kP}}{k}$$

The second may be integrated by parts and by use of the expression

$$\frac{d^2Q}{dP^2} = \frac{g}{u^2}$$

the following expression is obtained

$$Z_2 = - \frac{dQ}{dP} \frac{e^{-kP}}{k} + \int_0^P \frac{g}{u^2} \frac{e^{-kP}}{k} dP$$

where, as follows from Eq. (324)

$$\frac{dQ}{dP} = \frac{C}{\sigma a_o^2} \left[I(u/a_o) - I(u_o/a_o) \right] - k_o \int_0^P e^{-kP} \frac{g}{u^2} dP$$

The third integral Z_3 may be integrated by parts; it is

$$Z_3 = \left[P - \frac{k_o}{k} e^{-kP}\right] \int_0^P e^{-kP} \frac{g}{u^2} dP$$

$$- \int_0^P P e^{-kP} \frac{g}{u^2} dP + \frac{k_o}{k} \int_0^P e^{-2kP} \frac{g}{u^2} dP$$

With

$$Z_4 = \int_0^P e^{-kP} \frac{g}{u^2} dP$$

$$Z_5 = \int_0^P P e^{-kP} \frac{g}{u^2} dP$$

and

$$Z_6 = \int_0^P e^{-2kP} \frac{g}{u^2} dP$$

it follows that

$$\Delta Q_1 = \frac{k_o}{k} \left[\frac{C}{\sigma a_o^2} \left\{ I(u/a_o) e^{-kP} - I(u_o/a_o) \right\} - (1 + kp) Z_4 + kZ_5 - k_o Z_6 \right]$$

Z_4, Z_5, and Z_6 can be evaluated approximately by expanding $1/U^2$ in series. The first two terms of this series are

$$\frac{1}{U^2} = \frac{1}{U_o^2} + \frac{2}{U_o^3} G(U_o)(1 + k_o) \frac{\sigma}{C} P + \cdots$$

It follows that

$$\Delta Q_o = \frac{k_o}{k}\,\frac{C}{\sigma a_o^2}\left[I(u/a_o)\,e^{-kP} - I(u_o/a_o)\right]$$

$$-\frac{k_o g}{k^2 u_o^2}\left[kP + \frac{k_o}{2}\left(1 - e^{-2kP}\right)\right]$$

$$+\frac{2k_o g}{k^3 u_o^3}\,G(u_o/a_o)(1 + k_o)\,\frac{\sigma a_o}{C}\left[1 - \frac{k_o}{4} - kP\right.$$

$$\left. - e^{-kP}\left\{1 - \frac{k_o}{4}\,(1 + 2kP)\,e^{-kP}\right\}\right] \tag{340}$$

More terms in the series for $1/U^2$ may be retained if necessary. On the other hand, one would expect to drop some of the exponential terms when P is large.

3.3 A Windage-Jump Correction - A spinning projectile fired into a crosswind experiences an angular deflection of its direction of motion out of its initial plane of yaw. This deflection is numerically equal to about 5 percent of the initial angle of attack (yaw). For an initial yaw of 1^o = 17.5 milliradians, the windage jump is about 0.87 milliradians, or for 10^o yaw it is about 8.7 milliradians. For the 20-mm, M56 round, the windage jump in milliradians is, as a rule of thumb, numerically equal to the yaw angle in degrees (e.g., 5^o yaw, 5 milliradians windage jump). For forward fire from a fixed gun in an aircraft, windage jump is usually ignored since the dispersion of such systems is of the order of 5 milliradians. For a flexible, side-firing, gun system, the windage jump must be included, however, since the initial yaw may be as high as 20^o.

For aircraft fire, the initial plane of yaw is the plane containing the velocity vector of the aircraft $\vec{V}_A$ and the muzzle velocity of the gun $\vec{V}_M$. The initial yaw angle δ_o is the angle between $\vec{V}_M$ and the projectile initial velocity vector $\vec{u}_o$, where

$$\vec{u}_o = \vec{V}_A + \vec{V}_M$$

If the angle between $\vec{V}_A$ and $\vec{V}_M$ (the gun angle) is A, then, as can be seen from Fig. 17, δ_o is given by the relation

$$\delta_o \simeq \sin \delta_o = \frac{V_A \sin A}{u_o} \tag{341}$$

where

$$u_o = \sqrt{V_A^2 + V_M^2 + 2V_A V_M \cos A} \tag{342}$$

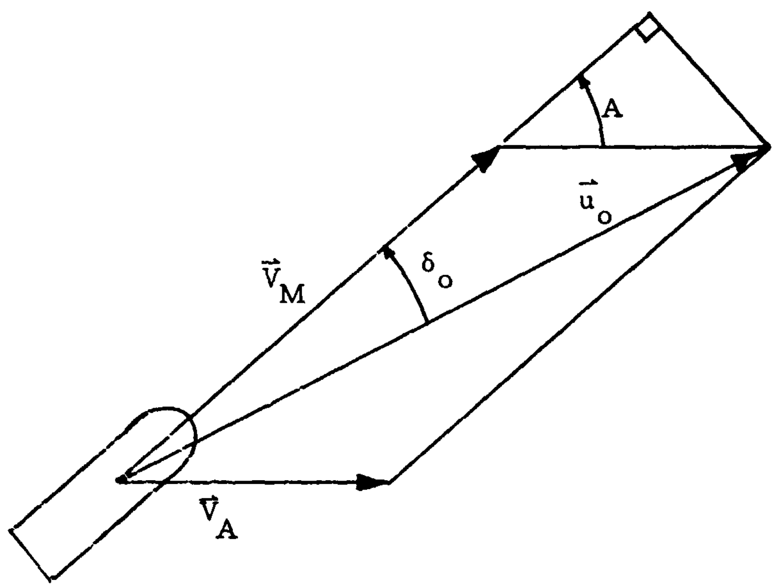


Figure 17

Projectile Geometry Showing Initial Angle of Attack δ_o in the Initial Plane of Yaw

The magnitude of the windage jump in radians is given approximately in Ref. 9 as

$$\epsilon = \frac{b\delta_o}{u_o} \tag{343}$$

where

$$b = \frac{AN}{md}\left(\frac{K_L}{K_M}\right) \tag{344}$$

A is the axial moment of inertia, N is the spin, m is the mass, d is the diameter, K_L is the lift coefficient, and K_M is the overturning moment coefficient. (Note that this expression does not contain the Magnus moment coefficients K_F and K_T.)

In calculations, the windage jump is usually treated as a small correction to $\vec{u}_o$, in which case $\vec{u}_o$ is defined as

$$\vec{u}_o = \vec{V}_A + \vec{V}_M + \vec{J} \tag{345}$$

where $J = u_o\epsilon$ and $\vec{J}$ is in tne direction of $\vec{V}_A \times \vec{V}_M$. From Eq. (341), Eq. (343), and since $|\vec{V}_A \times \vec{V}_M| = V_A V_M \sin A$, it follows that

$$\vec{J} = \frac{\left(\vec{V}_A \times \vec{V}_M\right)}{|\vec{V}_A \times \vec{V}_M|} u_o\epsilon \simeq \frac{b}{u_o V_M}\left(\vec{V}_A \times \vec{V}_M\right) \tag{346}$$

Since J is small, the magnitude of u_o is not changed significantly. The direction of $\vec{P}$ is taken to be along $\vec{u}_o$, so

$$\vec{P} = \frac{\vec{u}_o}{u_o} P \tag{347}$$

An alternate treatment of the windage jump can be based upon the set of approximate equations of motion of Section IV, or on any appropriate

six-degree-of-freedom code. The Eglin code R370, based upon the approximate equations, is used to calculate windage-jump components (among other things) for the 20-mm, M56 round in and perpendicular to the initial plane of yaw. These components may be used instead of $\vec{J}$ above. Experience indicates that these windage-jump components are, for large enough P, very nearly independent of all initial condition parameters except δ_o. Appropriate formulae and necessary explanations follow.

Windage-jump parameters are J_2 and J_3. J_3 corresponds to $\vec{J}$ above and is normal to the plane of yaw, whereas J_2 is in the plane of yaw and is zero in the elementary theory. J_2 and J_3 are output in milliradians as a function of time but they approach constant values as time increases. From J_2 and J_3, swerve components in the units of P (distance) are calculated from

$$S_2 = 10^{-3} J_2 P \tag{348}$$

$$S_3 = 10^{-3} J_3 P \tag{349}$$

These components are transformed into an ξ, η, ζ coordinate system (Fig. 12) by means of the equations (from Section IV)

$$S_\xi = -(S_2 \cos \phi_o - S_3 \sin \phi_o) \sin \theta_o \tag{350}$$

$$S_\eta = (S_2 \cos \phi_o - S_3 \sin \phi_o) \cos \theta_o \tag{351}$$

$$S_\zeta = S_2 \sin \phi_o + S_3 \cos \phi_o \tag{352}$$

where θ_o is the elevation angle of $\vec{u}_o$ above the horizontal and ϕ_o is the angle measured about $\vec{u}_o$ between the vertical plane and the initial position of the plane of yaw. Components of ξ, η, and ζ are given by

$$\xi = P \cos \theta_o + S_\xi \tag{353}$$

$$\eta = P \sin \theta_o - Q + S_\eta \tag{354}$$

$$\zeta = S_\zeta \tag{355}$$

The ξ, η, ζ coordinate system is shown in Fig. 12; ξ and ζ are horizontal, and η is vertical and is measured up. The system is right-handed, so ζ is measured out of the page. $\vec{P}$, $\vec{Q}$, and $\vec{u}_o$ are in the ξ, η plane. Expressions for θ_o and ϕ_o are derived in Section VI.

Values of J_2 and J_3 used in 20-mm, M56 projectile calculations are

$$J_2 = 5\delta_o \tag{356}$$

$$J_3 = 53\delta_o \tag{357}$$

where δ_o is in radians, and J_2 and J_3 are in milliradians.

3.4 Independence of Correction Terms - The yaw-drag and the variable air-density corrections were derived as if the two effects are independent. This has some physical justification since the yaw-drag effect is important at the beginning of the trajectory, whereas the change in air density is greatest (e.g., in the air-to-ground case) at longer ranges. There is a difficulty, however, with the equation containing P. The question arises as to whether Eq. (314) or Eq. (337) should be used. This is resolved by including both effects in Eq. (289). The following relation is obtained

$$-\frac{C}{\sigma}\frac{dU}{G(U)} = e^{-hP \sin \theta_o}\left[1 + k_o e^{-kP}\right] dP$$

$$\simeq \left[e^{-hP \sin \theta_o} + k_o e^{-kP}\right] dP$$

This is a good approximation since in practice $k >> h$. (For the 20-mm, M56 round, $k \approx 0.004$, whereas $h \simeq 0.00003$.) It follows that the correct expression to be used must be

$$\frac{C}{\sigma}\left[S(u/a_o) - S(u_o/a_o)\right] = \frac{1 - e^{-hP\sin\theta_o}}{h\sin\theta_o} + k_o\frac{1 - e^{kP}}{k}$$

$$\simeq P\left\{1 - \frac{hP\sin\theta_o}{2}\right\} + \frac{k_o}{k} \qquad (358)$$

This last approximation should be good past $P \simeq 1,000$ ft.

3.5 Summary of Results - Formulae derived in this section are listed below for convenience. Q and t denote gravity drop and time of flight as obtained from the basic Siacci theory, whereas Q_c and t_c denote corrected values.

$$S(u/a_o) = S(u_o/a_o) + \frac{k_o}{2cC} + \frac{\sigma}{C}P\left\{1 - \frac{hP\sin\theta_o}{2}\right\}$$

$$t = \frac{C}{\sigma a_o}\left[T(u/a_o) - T(u_o/a_o)\right]$$

$$Q = \left(\frac{C}{\sigma a_o}\right)^2\left[A(u/a_o) - A(u_o/a_o) - I(u_o/a_o)\frac{\sigma}{C}P\right]$$

$$t_c = t + \Delta t_\rho + \Delta t_1$$

$$Q_c = Q + \Delta Q_\rho + \Delta Q_1$$

$$\Delta t_\rho = \left(\frac{C}{\sigma a_o}\right)^2 a_o h\sin\theta_o\left\{\left[H(u/a_o) - H(u_o/a_o)\right] - S(u_o/a_o)\frac{\sigma a_o}{C}t\right\}$$

$$\Delta t_1 = \frac{k_o}{ku_o}\left[1 + \frac{\sigma a_o}{C}\,\frac{1 + k_o}{ku_o}\,G(u_o/a_o)\right]$$

$$\Delta Q_1 = -\frac{k_o}{k}\ \frac{C}{\sigma a_o^2}\ I_o(u_o/a_o)$$

$$-\frac{k_o g}{k^2 u_o^2}\left[\frac{k_o}{2} + kP - 2\frac{\sigma a_o}{C}\ \frac{1+k_o}{k u_o}\ G(u_o/a_o)\left\{1 - \frac{k_o}{4} - kP\right\}\right]$$

$$H(U) = \int_U^V \frac{S(U)\,dU}{UG(U)}$$

$$\Delta Q_\rho = \left(\frac{C}{\sigma a_o}\right)^3 a_o h \sin\theta_o \left[W_1(u/a_o) - W_1(u_o/a_o)\right.$$

$$- 2S(u_o/a_o)\left\{A(u/a_o) - A(u_o/a_o)\right\}$$

$$- I(u_o/a_o)\left\{W_2(u/a_o) - W_2(u_o/a_o)\right\}$$

$$\left. + \left\{A(u_o/a_o) + I_o(u_o/a_o)\,S(u_o/a_o)\right\}\left\{S(u/a_o) - S(u_o/a_o)\right\}\right]$$

$$W_1(U) = \int_U^V \frac{2S(U)\,I(U) - A(U)}{G(U)}\,dU$$

$$W_2(U) = \int_U^V \frac{S(U)}{G(U)}\,dU$$

Exponential terms have been eliminated from the correction terms for Δt_1 and ΔQ_1. Additional equations (Sterne's theory) are

$$k = 2\sigma c$$

$$c = c' + \frac{c''}{s_o - 1}$$

$$c' = \frac{\rho_o d^2}{2m}\left[\frac{md^2}{B} K_H + K_L\right]$$

$$c'' = \frac{\rho_o d^2}{2m} K_D$$

$$s_o = \frac{A^2 N^2}{4B\ \rho\, d^3 u_o^2 K_M}$$

$$k_o = \frac{s_o - 1/2}{s_o - 1} K_{D_{\delta^2}} \delta_o^2$$

and also

$$h = 3.158 \times 10^{-5}\ \text{ft}^{-1}$$

An alternate expression for c derived from the approximate theory is

$$c = \frac{\rho_o d^2}{m}\left[K_L - \frac{md^2}{A} K_T\right]$$

Equations describing the windage jump are

$$\vec{u}_o = \vec{V}_A + \vec{V}_M + \vec{J}$$

$$\vec{J} = \frac{b}{u_o V_M}\left(\vec{V}_A \times \vec{V}_M\right)$$

$$b = \frac{AN}{md}\left(\frac{K_L}{K_M}\right)$$

$$\vec{P} = \frac{\vec{u}_o}{u_o} P$$

Alternate windage-jump equations from the approximate theory (see Fig. 12) are

$$S_2 = 10^{-3} J_2 P$$

$$S_3 = 10^{-3} J_3 P$$

$$S_\xi = -(S_2 \cos \phi_o - S_3 \sin \phi_o) \sin \theta_o$$

$$S_\eta = (S_2 \cos \phi_o - S_3 \sin \phi_o) \cos \theta_o$$

$$S_\zeta = S_2 \sin \phi_o + S_3 \cos \phi_o$$

$$\xi = P \cos \theta_o + S_\xi$$

$$\eta = P \sin \theta_o - Q + S_\eta$$

$$\zeta = S_\zeta$$

SECTION VI

AIRBORNE FIRE CONTROL APPLICATIONS

1. General

In the present application, the gun-pointing problem is complicated by the fact that both the gun and the target may be moving. In the air-to-ground application, the target is stationary in many instances, but otherwise its motion often should be considered. In a situation where the projectile time of flight is 1 sec, for example, the distance moved by a 60-mph vehicle is 88 ft. The problem at hand, the fire control problem, involves predicting the motion of the target so that the hit position may be obtained, and involves determination of the correct gun-pointing direction to score a hit on the target. Prediction of target motion is known as kinematic prediction, whereas the gun-pointing problem is known as ballistic prediction. Kinematic and ballistic prediction are discussed briefly in Subsection 2.

In Sections IV and V, equations of motion were developed for a projectile in flight, but initial conditions were not considered. Initial conditions will be developed in Subsection 4 for the equations of motion developed in Sections IV and V.

Of interest in current applications is the gatling gun. The specification of initial conditions for a projectile fired from such a weapon mounted in a turret in a moving aircraft is complicated. The combined motions of the rotating barrel cluster, the turret, and the aircraft can cause errors in calculations if they are not accounted for. For example, the distance from the aircraft center of mass to the gun muzzle may be, say, 10 ft, and the aircraft angular velocity perpendicular to this distance may be, say, 60 deg/sec or about 1 rad/sec. The component of projectile velocity due to this angular rotation is, then, 10 ft $\times$ 1 rad/sec = 10 ft/sec. If this component is perpendicular to the muzzle velocity, which is about 3300 ft/sec for the 20-mm, M56 round, the angular error due to ignoring this angular motion is 10 ft/sec divided by 3300 ft/sec

and this equals .003 rad or 3 mr. Slew rates of the gun will probably be of the order of 1 rad/sec and the distance from the gimbals to the gun muzzle may be between 5 and 10 ft. The error due to this cause may be up to 3 mr also. The M61 gatling gun fires up to 6000 rounds per minute from six barrels, so the barrel cluster spins at 1000 rotations per minute or $2\pi \times 1000/60 = 105$ rad/sec. The distance from the barrel-cluster axis of rotation and the center of any barrel is 1.877 in. It follows that the rotating barrel cluster imparts a velocity of 105 rad/sec times 1.887/12 ft equals 16.4 ft/sec to the projectile. This amounts to $1000 \times 16.4/3300 = 5$ mr. Evidently these effects should be accounted for in initial condition calculations and Subsection 3 contains the necessary coordinate transformations.

2. Kinematic and Ballistic Prediction

The fire control problem may be divided into two parts: kinematic prediction and ballistic prediction. To determine the future target path as a function of time and to find the hit position, given the time of flight, is the kinematic prediction problem. For the present purposes, it will be assumed that the target path is a known function of time. Kinematic prediction in air-to-air applications is the subject of a separate study.

The determination of the correct gun-pointing direction to score a hit on the target, given the hit position, is the ballistic prediction problem. When the gun and/or the target is moving, hits cannot be scored by pointing the gun directly at the target except in unusual circumstances. The correct gun-pointing direction is found by an iterative trial-and-error procedure. A first guess is made for the gun-pointing direction, a trajectory is calculated, and the miss distance is used to correct the gun-pointing direction. If a second calculated trajectory misses the target, the process is repeated. The iterative procedure is continued until the gun is on target.

Solution of the ballistic prediction problem, as described, implies knowledge of the hit position, which in turn implies knowledge of the time of flight. But the time of flight to any point is not known until the

correct gun-pointing direction to score a hit on that point is determined and a trajectory has been calculated. In other words, the kinematic and ballistic problems are interdependent and cannot be solved separately.

An iterative solution is available: a first estimate to the time of flight is chosen, and, given the target path, an estimated hit position is calculated. Given the estimated hit position, the gun-pointing direction to score a hit at that point is determined, as described above, and the time of flight is calculated. The calculated time of flight and the first estimate to the time of flight are used to correct the estimated hit position and the process is repeated until the correct hit position is obtained.

The iterative solution of the kinematic and ballistic prediction problems, as described, requires the calculation of several trajectories. The best such procedure will keep the required number of trajectory computations to a minimum. An investigation of algorithms for onboard kinematic and ballistic prediction is needed, and will not be treated here. The purpose of this subsection is to point out the existence of this problem.

3. Coordinate Systems and Transformations

In the development of initial-condition equations for a projectile fired from a turreted gatling gun, five right-handed, rectilinear coordinate systems will be used and they are as follows: (1) an earth-fixed, inertial system, S_I; (2) a system, S_A, fixed in the aircraft with its origin at the center of mass, with the x_A axis along the body longitudinal axis and directed out the nose, with the y_A axis out the right wing, and with the z_A axis pointed down toward the aircraft floor; (3) a system, S_T, fixed in the aircraft but with its origin on the axis of the outer turret gimbal; (4) a system, S_G, attached to the gun frame with its origin on the axis of the inner gimbal and which rotates about the two gimbal axes; and (5) a system, S_B, attached to the barrel cluster with its origin on the gatling gun barrel-cluster axis of rotation.

Coordinate transformations between these systems are needed so the projectile velocity, $\vec{u}_o$, angular velocity, $\vec{\omega}_o$, position, $\vec{R}_o$, yaw, δ_o,

and precession angle, ϕ_o, may be calculated in inertial space, S_I, at firing time. To calculate δ_o and ϕ_o, the direction of the longitudinal axis of the projectile, i.e., the boreline of the gun barrel, is needed in inertial space.

To expedite setting up these transformations, matrix notation will be used. The appropriate derivations of matrix equations are given in the appendix. A summary of results follows.

3.1 Matrix Notation - The results of the appendix may be summarized as follows: The transformation of the coordinates of a point P as observed in space S' into the coordinates of P as observed in space S is given by the matrix equation

$$a = c + T a'$$

where a' represents P as observed in S', the symbol a represents P in S, c is the position of the origin of S' as observed in S, and T is a 3×3 matrix which relates the angular orientation of S' to that of S. The transformation of the velocity, $\dot{a}'$, of P as observed in S' into the velocity, $\dot{a}$, of P as observed in S is given by

$$\dot{a} = \dot{c} + T(\dot{a}' + \Omega' a')$$

where $\dot{c}$ is the velocity of the origin of S' as observed in S, and Ω' is the 3 × 3 matrix representation of the angular velocity of S' relative to S. Ω' may be written as

$$\Omega' = \begin{pmatrix} 0, & -\Omega_{z'}, & \Omega_{y'} \\ \Omega_{z'}, & 0, & -\Omega_{x'} \\ -\Omega_{y'}, & \Omega_{x'}, & 0 \end{pmatrix}$$

where $\Omega_{x'}$, $\Omega_{y'}$, and $\Omega_{z'}$ are the components of angular velocity of the S' system with respect to the S system as measured in the S' system. Ω' may be transformed into the S system by the relation

$$\Omega = T\Omega' \tilde{T}$$

where $\tilde{T}$ is the transpose of T. The column vector representation of Ω' is

$$\omega' = \begin{pmatrix} \Omega_{x'} \\ \Omega_{y'} \\ \Omega_{z'} \end{pmatrix}$$

however, and the transformation from S' to S is

$$\omega = T\omega'$$

The column vector notation is obviously easier to use in coordinate transformations.

A proof that angular velocities add is also included in the appendix. For example, if ω' is the angular velocity of S' with respect to S as measured in S', and ω^* is the angular velocity of a solid body measured in S', then

$$\omega = T(\omega' + \omega^*)$$

is the angular velocity of the solid body measured in S.

3.2 <u>Subscript Convention</u> - It will be convenient to employ a subscript notation to designate coordinate systems. Thus, the coordinate space J is designated by S_J and the coordinate space K is S_K. The angular orientation of the coordinate axes of S_J with respect to S_K is given by

$$e_J = e_K T_{KJ}$$

or, equivalently, the angular orientation of the coordinate axes of S_K with respect to S_J is

$$\tilde{e}_K = T_{KJ} \tilde{e}_J$$

where e_J and e_K are row matrices composed of the unit vectors directed along the axes of the two systems. The transformation from S_J to S_K is

$$b_K = b_{KJ} + T_{KJ} b_J$$

where b_J is the vector (column matrix) representing the coordinates of a point P in S_J and b_K is the vector representing P in S_K. The vector b_{KJ} represents the distance, measured in S_K, from the origin, O_K, of S_K to the origin, O_J, of S_J.

The velocity transformation is given by

$$V_K = V_{KJ} + T_{KJ} (V_J + \Omega_{KJ} b_J)$$

where $V_J = \dot{b}_J$ and $V_K = \dot{b}_K$ represent the velocity of P in S_J and S_K, respectively. $V_{KJ} = \dot{b}_{KJ}$ is the velocity of O_J with respect to O_K as measured in S_K, and Ω_{KJ} is the angular velocity of S_J with respect to S_K as measured in S_J. The column matrix representation of Ω_{KJ} is ω_{KJ}.

3.3 Coordinate Transformations - Coordinate systems and transformations are described, starting with S_B as shown in Fig. 38. There is some freedom of choice in the way these systems may be defined, so definitions were made as convenient.

In S_B, the barrel-cluster system, the x_B axis is along the barrel-cluster rotation axis, and the gun muzzle, for any particular barrel at the time of firing, lies on the y_B axis a distance L from the origin.

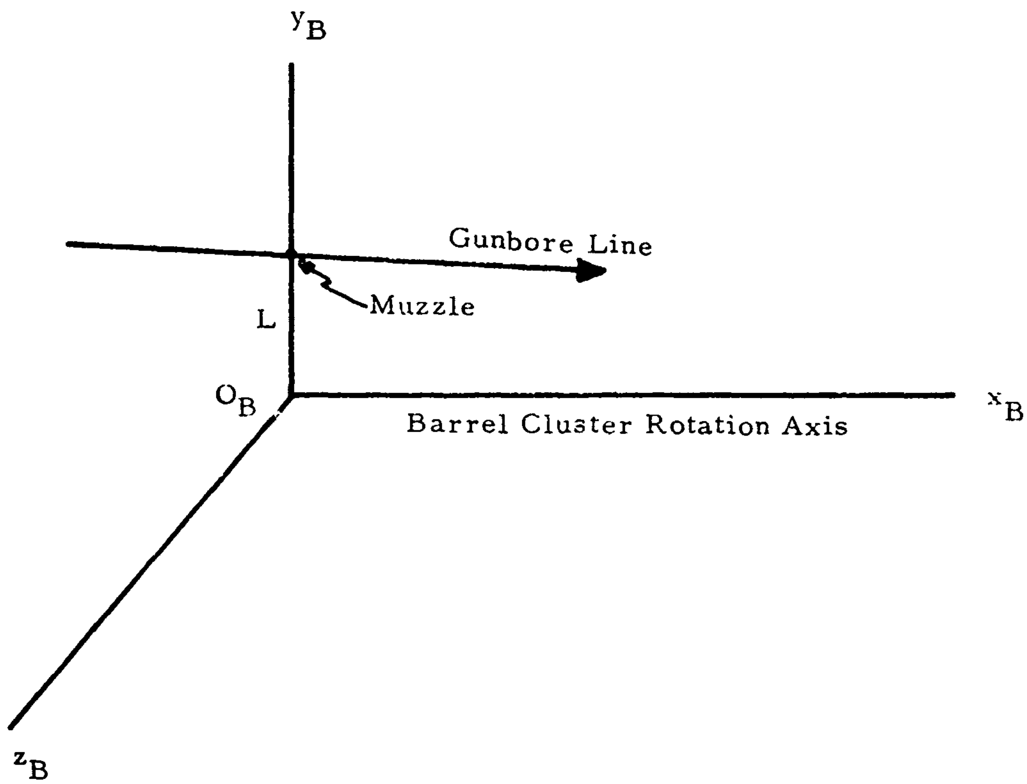


Figure 18

Barrel Cluster System, S_B

The position of the gun muzzle at the time of firing is given by the matrix

$$b_B = \begin{pmatrix} 0 \\ L \\ 0 \end{pmatrix} \tag{359}$$

and the direction of the gunbore axis is specified by the matrix

$$\nu_B = \begin{pmatrix} \ell_B \\ m_B \\ n_B \end{pmatrix} \tag{360}$$

The velocity of the projectile as measured in S_B is parallel to ν_B and is given by

$$u_B = \nu_B V_M \tag{361}$$

where V_M is the muzzle velocity (a scalar), and the angular velocity of the projectile as measured in S_B is

$$\omega_B = \frac{2\pi}{nd} V_M \nu_B \tag{362}$$

where n is the distance traveled in units of d during one complete rotation of the projectile. Units of n are calibers/turn. It is our purpose to transform these four vectors into inertial space, S_I.

The gun system, S_G, is shown in Fig. 19, where the x_G axis is parallel to x_B and the z_G axis lies along the inner gimbal axis. The distance from the origin of S_G to the origin of S_B is represented by the matrix

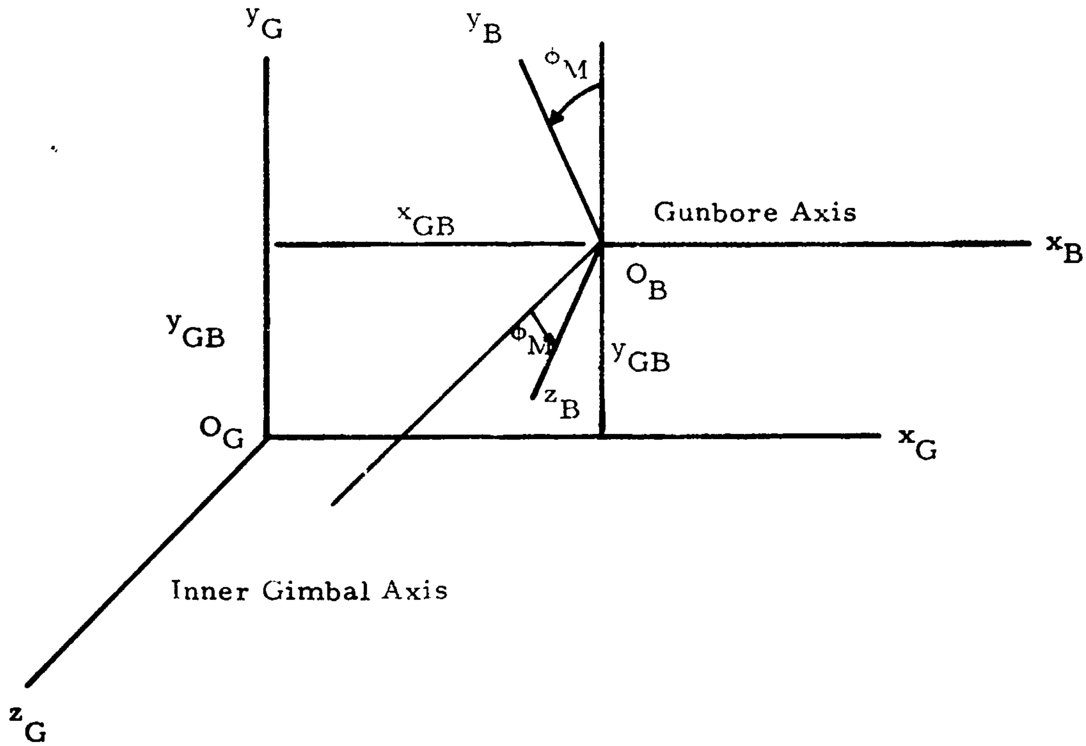


Figure 19

Relation of Gun Space, S_G, and Barrel Cluster Space, S_B

$$b_{GB} = \begin{pmatrix} x_{GB} \\ y_{GB} \\ 0 \end{pmatrix} \tag{363}$$

The transformation matrix between S_B and S_G is

$$T_{GB} = \begin{pmatrix} 1, & 0, & 0 \\ 0, & \cos\phi_M, & -\sin\phi_M \\ 0, & \sin\phi_M, & \cos\phi_M \end{pmatrix} \tag{364}$$

where ϕ_M is the angle between y_B and the x_G, y_G plane at the time of fire, and the position of the gun muzzle as measured in S_G is

$$b_G = b_{GB} + T_{GB}\, b_B \tag{365}$$

The direction of the gunbore axis as measured in S_G is

$$\nu_G = T_{GB}\, \nu_B \tag{366}$$

and the angular velocity of the projectile in S_G is

$$\omega_G = T_{GB}\,(\omega_{GB} + \omega_B) \tag{367}$$

where

$$\omega_{GB} = \begin{pmatrix} \dot{\phi}_M \\ 0 \\ 0 \end{pmatrix} \tag{368}$$

The projectile velocity in S_G is

$$u_G = T_{GB}\left(u_B + \Omega_{GB} b_B\right) \tag{369}$$

where Ω_{GB} is the 3×3 matrix representation of ω_{GB};

$$\Omega_{GB} = \begin{pmatrix} 0 \ , & 0 \ , & 0 \\ 0 \ , & 0 \ , & -\dot{\phi}_M \\ 0 \ , & \dot{\phi}_M, & 0 \end{pmatrix} \tag{370}$$

The relation of the gun system, S_G, with respect to the turret system, S_T, is shown in Fig. 20. The distance from the origin of S_T to that of S_G is

$$b_{TG} = \begin{pmatrix} r_{TG} \cos A' \\ 0 \\ r_{TG} \sin A' \end{pmatrix} \tag{371}$$

The transformation matrix relating vectors in S_G and S_T is

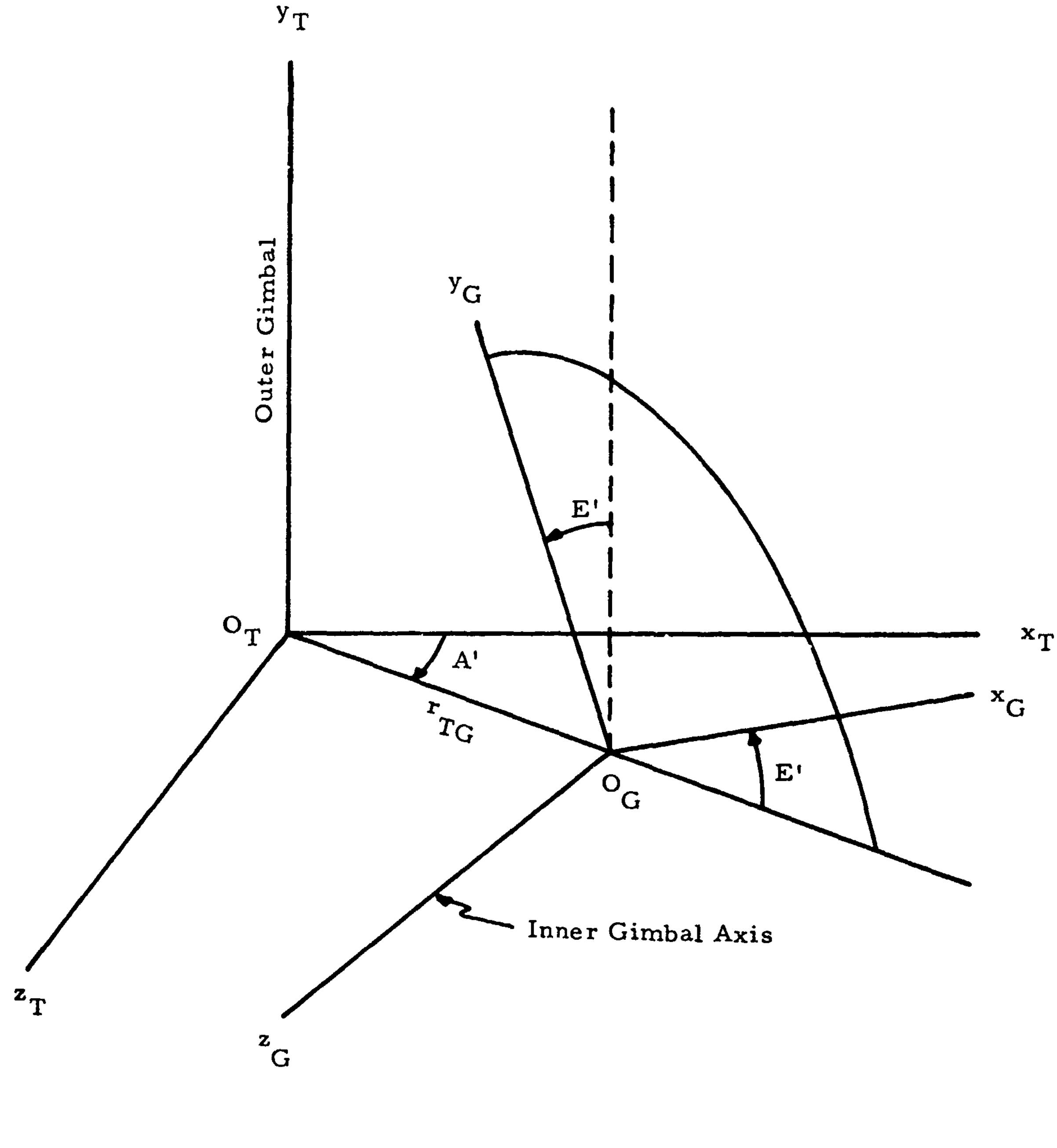


Figure 20
Relation of Turret Space, S_T, and Gun Space, S_G.
O_G is in the x_T, z_T Plane

$$
T_{TG} = \begin{pmatrix} \cos E' \cos A' , & -\sin E' \cos A' , & -\sin A' \\ \sin E' , & \cos E' , & 0 \\ \cos E' \sin A' , & -\sin E' \sin A' , & \cos A' \end{pmatrix} \tag{372}
$$

and the position of the gun muzzle as measured in S_T is

$$
b_T = b_{TG} + T_{TG} b_G \tag{373}
$$

The direction of the gunbore as measured in S_T is

$$
\nu_T = T_{TG} \nu_G \tag{374}
$$

and the angular velocity of the projectile in S_T is

$$
\omega_T = T_{TG} (\omega_{TG} + \omega_G) \tag{375}
$$

where

$$
\omega_{TG} = \begin{pmatrix} -\dot{A}' \sin E' \\ -\dot{A}' \cos E' \\ \dot{E}' \end{pmatrix} \tag{376}
$$

in S_G coordinates. The projectile velocity in S_T is given by

$$
u_T = u_{TG} + T_{TG} (u_G + \Omega_{TG} b_G) \tag{377}
$$

where

$$u_{TG} - \dot{b}_{TG} = \dot{A}' \begin{pmatrix} -r_{TG} \sin A' \\ 0 \\ r_{TG} \cos A' \end{pmatrix} \quad (378)$$

and

$$\Omega_{TG} = \begin{pmatrix} 0 & , & -\dot{E}' & , & -\dot{A}' \cos E' \\ \dot{E}' & , & 0 & , & \dot{A}' \sin E' \\ \dot{A}' \cos E', & & -\dot{A}' \sin E', & & 0 \end{pmatrix} \quad (379)$$

Since both aircraft space, S_A, and turret space, S_T, are fixed in the aircraft, the position of the gun muzzle in S_A is

$$b_A = b_{AT} + T_{AT} b_T \quad (380)$$

Whereas it is possible to define T_{AT} in terms of orientation angles relating S_T and S_A, this will serve no particular purpose here and will be omitted. T_{AT} can be defined in any particular application when the need arises. The vector b_{AT} is the distance from the aircraft center of gravity to the origin of S_T and is measured in S_A. The direction of the gunbore in S_A is

$$\nu_A = T_{AT} \nu_T \quad (381)$$

The angular velocity of the projectile in S_A is

$$\omega_A = T_{AT} \omega_T \quad (382)$$

and the projectile velocity in S_A is

$$u_A = T_{AT}\, u_T \tag{383}$$

The position of the gun muzzle in inertial space, S_I, is

$$b_I = b_{IA} + T_{IA}\, b_A \tag{384}$$

where b_{IA} is the position of the aircraft center of mass in S_I, and T_{IA} is the matrix which relates the relative orientations of S_I and S_A. Writing T_{IA} in terms of orientation angles of the aircraft would serve no purpose here and will be omitted. The direction of the gunbore in S_I is

$$\nu_I = T_{IA}\, \nu_A \tag{385}$$

and the angular velocity of the projectile in S_I is

$$\omega_I = T_{IA}\,(\omega_{IA} + \omega_A) \tag{386}$$

where ω_{IA} is the angular velocity of S_A with respect to S_I as measured in S_A and

$$\omega_{IA} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \tag{387}$$

The velocity of the projectile in S_A is

$$u_I = u_{IA} + T_{IA}\,(u_A + \Omega_{IA}\, b_A)$$

where

$$\Omega_{IA} = \begin{pmatrix} 0 , & -r , & q \\ r , & 0 , & -p \\ -q , & p , & 0 \end{pmatrix} \tag{388}$$

Parameters p, q, and r are the usual symbols for the components of aircraft angular velocity in the x_A, y_A, and z_A directions, respectively, and are assumed to be known. The matrix u_{IA} is equivalent to the aircraft velocity vector $\vec{V}_A$, i.e.,

$$\vec{u}_{IA} = \vec{V}_A \tag{389}$$

Usually, the aircraft position matrix, b_{IA}, is of no interest (only relative target position is needed) and the origin of S_I is taken to be at the instantaneous position of the aircraft center of mass at the time of fire. Thus

$$b_{IA} = 0 \tag{390}$$

and b_I is the location of the gun muzzle with respect to the aircraft center of mass. The aircraft altitude is, of course, assumed to be known. The symbol used in the next subsection for the initial bullet position at the time of fire is $\vec{R}_o$, so

$$\vec{R}_o = \vec{b}_I \tag{391}$$

Also the projectile velocity and angular velocities at time of fire are, respectively,

$$\vec{u}_o = \vec{u}_I \tag{392}$$

and

$$\vec{\omega}_o = \vec{\omega}_I \tag{393}$$

and the direction of the gunbore or of the shell longitudinal axis is

$$\vec{e}_o = \vec{\nu}_i \tag{394}$$

4. Initial Conditions

In Subsection 3, coordinate systems and transformations were introduced for the calculation of projectile velocity, $\vec{u}_o$, projectile angular velocity, $\vec{\omega}_o$, gun muzzle position, $\vec{R}_o$, and boreline direction, $\vec{e}_o$, in inertial space at the time of fire. It will be assumed that these vectors are available (defined in inertial space) so that appropriate initial conditions may be derived for trajectory computations using the equations of Sections IV and V.

4.1 Initial Conditions for the Matrix Formulation - Required initial conditions for use with the matrix formulation of the equations of motion, Section IV, Subsection 2, are values at time t_o for the components of $\vec{u}$, $\vec{\omega}$, $\vec{R}_o$, and the direction cosine matrix

$$\Lambda = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

By definition,

$$\vec{x}_1 = \vec{e}_o$$

and from Eq. (159) it is seen that initial values of ℓ_1, ℓ_2, and ℓ_3 are defined. Also, initial position $\vec{R}_0$ is given. Other initial parameters depend upon the choice of the x_2 and x_3 directions of the x_1, x_2, x_3 coordinate system and a convenient initial orientation is that shown in Fig. 21. It is seen that

$$\begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\beta'\cos\alpha', & \cos\beta'\sin\alpha', & \sin\beta' \\ -\sin\beta'\cos\alpha', & -\sin\beta'\sin\alpha', & \cos\beta' \\ \sin\alpha', & -\cos\alpha', & 0 \end{pmatrix} \begin{pmatrix} \vec{X} \\ \vec{Y} \\ \vec{Z} \end{pmatrix}$$

Comparison of this equation with Eq. (159) shows that

$$\ell_1 = \cos\beta'\cos\alpha'$$

$$\ell_2 = \cos\beta'\sin\alpha'$$

$$\ell_3 = \sin\beta'$$

and it follows that

$$\cos\beta' = \sqrt{1 - \ell_3^2}$$

$$\cos\alpha' = \frac{\ell_1}{\sqrt{1 - \ell_3^2}}$$

$$\sin\alpha' = \frac{\ell_2}{\sqrt{1 - \ell_3^2}}$$

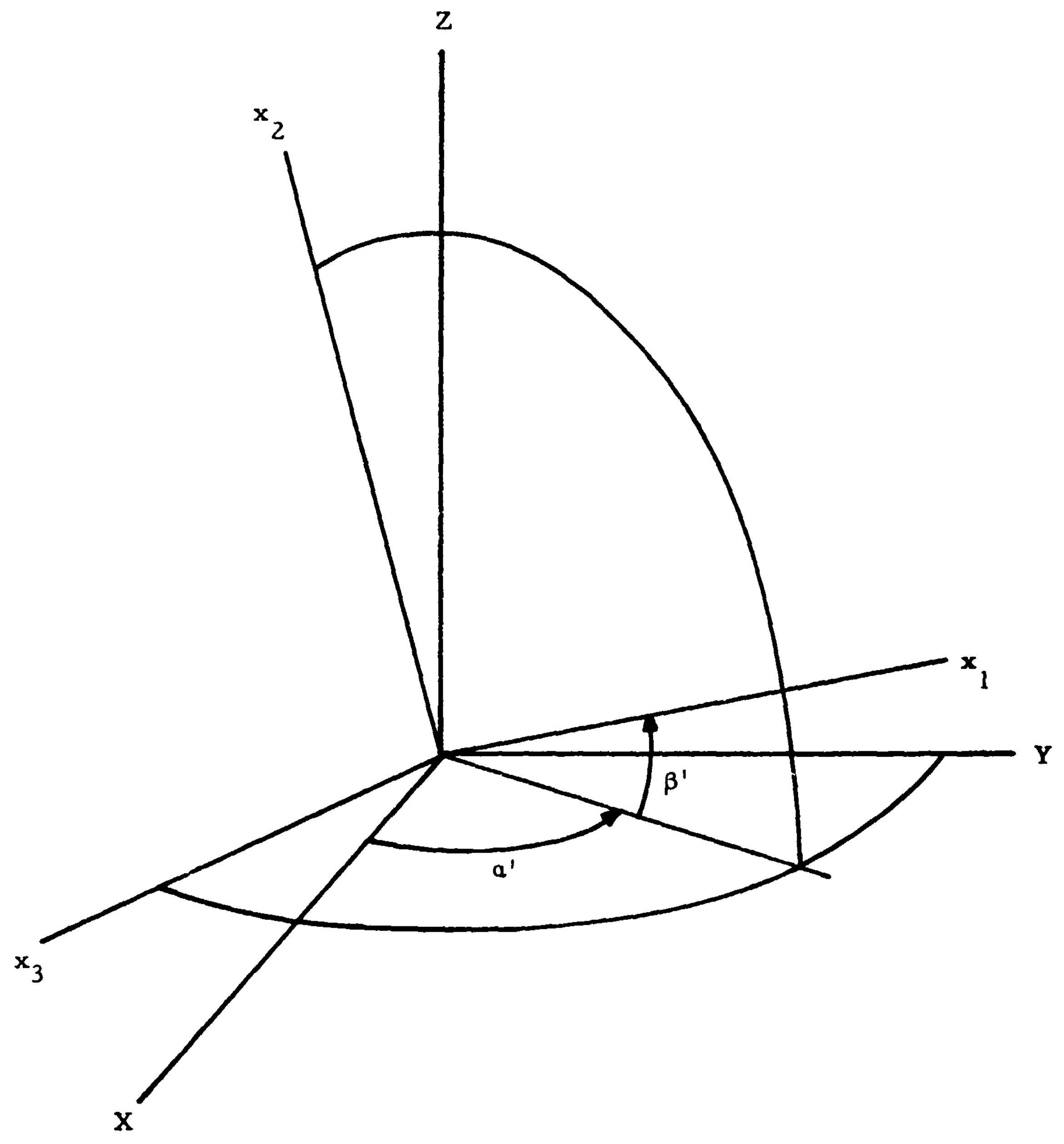


Figure 21
Relation between the x_1, x_2, x_3 System and the X, Y, Z System

Given α' and β', the other direction cosines may be calculated and so Λ is defined. Components of $\vec{u}_o$ and $\vec{\omega}_o$ in the x_1, x_2, x_3 coordinate system are given by the matrices Λu_o and $\Lambda\omega_o$, respectively, where u_o and ω_o are the inertial space matrix representations of $\vec{u}_o$ and $\vec{\omega}_o$, respectively. This completes the set of initial conditions for the matrix equation formulation of the six-degree-of-freedom equations.

4.2 Initial Conditions for the Euler Angle Formulation - Initial conditions for the Euler angle formulation of Section IV, Subsection 3 are specified by values for the following parameters: α_o, θ_o, u_o, ω_{Ao}, ω_{Bo}, ω_{3o}, δ_o, ϕ_o, ξ_o, η_o, and ζ_o. For convenience, the origin of the inertial space, S_I, is taken at the instantaneous position of the gun muzzle at the time of fire, so initial position coordinates ξ_o, η_o, and ζ_o are zero. As was explained in Section IV, the ξ, η, ζ axes are defined such that η is vertical, positive up, and the ξ axis lies along the projection of $\vec{u}_o$ in the horizontal plane. If the x_I, y_I, z_I axes are defined as shown in Fig. 22, the relation between the two systems is given by

$$\begin{pmatrix} \vec{\xi} \\ \vec{\eta} \\ \vec{\zeta} \end{pmatrix} = \begin{pmatrix} \sin B', & \cos B', & 0 \\ 0 \;\;, & 0 \;\;, & 1 \\ \cos B', & -\sin B', & 0 \end{pmatrix} \begin{pmatrix} \vec{x}_I \\ \vec{y}_I \\ \vec{z}_I \end{pmatrix} \tag{395}$$

In this equation, an arrow over a symbol denotes a unit vector in the direction associated with the symbol. If u_x, u_y, and u_z are the components of $\vec{u}_o$ in the x_I, y_I, and z_I directions, respectively,

$$u_o = \sqrt{u_x^2 + u_y^2 + u_z^2} \tag{396}$$

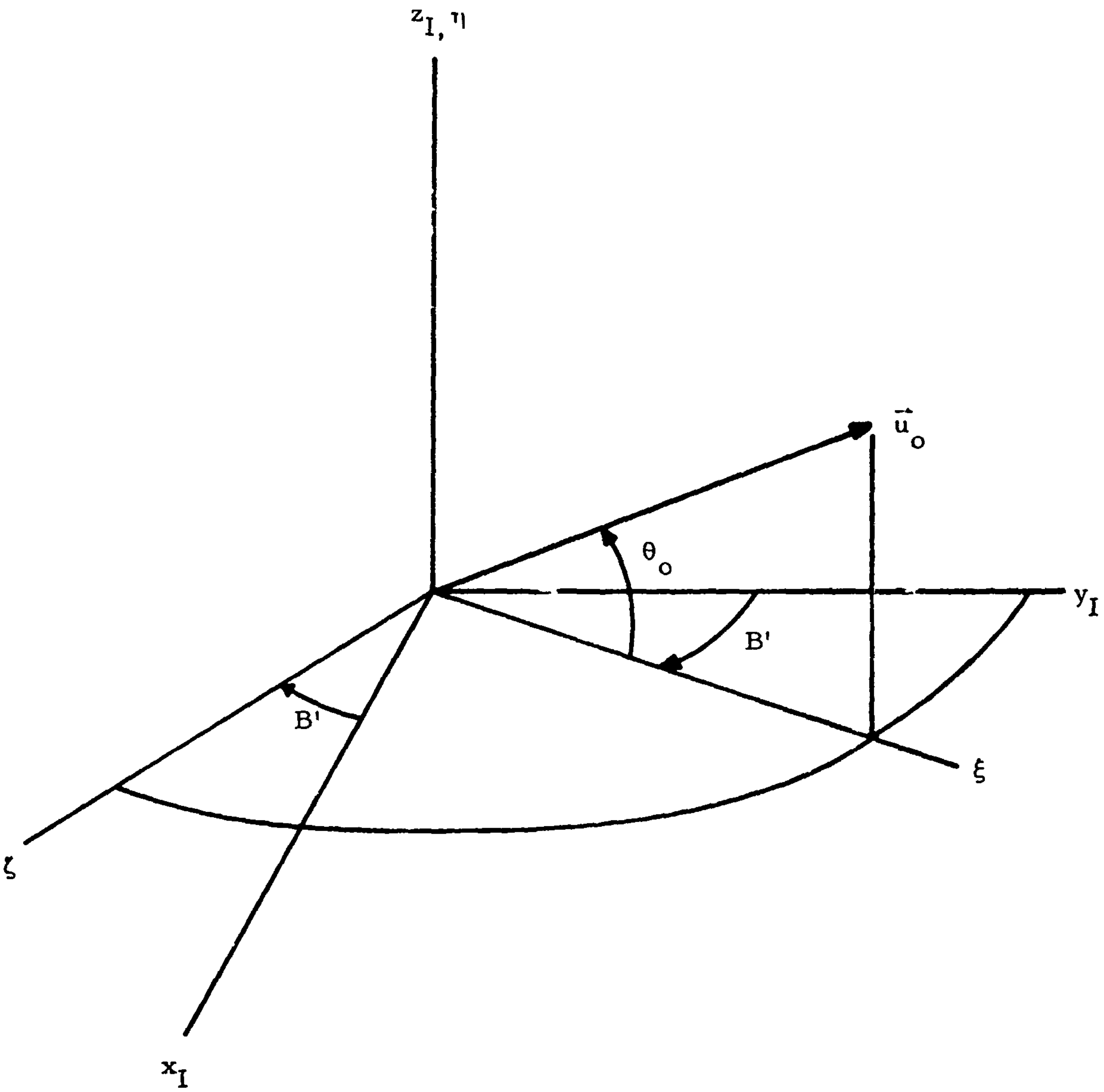


Figure 22

Relation of the ξ, η, ζ System with Respect to the x_I, y_I, z_I System and $\vec{u}_o$

$$\sin\theta_o = \frac{u_z}{u_o} \tag{397}$$

$$\cos\theta_o = \frac{u_{xy}}{u_o} \tag{398}$$

where

$$u_{xy} = \sqrt{u_x^2 + u_y^2} \tag{399}$$

and

$$\sin B' = \frac{u_x}{u_{xy}} \tag{400}$$

$$\cos B' = \frac{u_y}{u_{xy}} \tag{401}$$

By definition, $\alpha_o = 0$. If the components of $\vec{e}_o$ in the x_I, y_I, z_I system are e_x, e_y, and e_z,

$$\cos\delta_o = \frac{\vec{u}_o \cdot \vec{e}_o}{u_o} = \frac{1}{u_o}\left(u_x e_x + u_y e_y + u_z e_z\right) \tag{402}$$

and

$$\sin\delta_o = \sqrt{1 - \cos^2\delta_o} \tag{403}$$

An expression for the angle ϕ_o can be obtained with the aid of Fig. 23. In the figure, η' is in the vertical plane and is perpendicular to $\vec{u}_o$. Hence, ϕ_o is the angle between η' and the projection, $\vec{e}_\perp$, of $\vec{e}_o$ in the η', ζ plane. From Eq. (395), the components of $\vec{e}_o$ in the ξ, η, ζ coordinate system are

$$e_\xi = e_x \sin B' + e_y \cos B' \tag{404}$$

$$e_\eta = e_z \tag{405}$$

$$e_\zeta = e_x \cos B' - e_y \sin B' \tag{406}$$

Hence,

$$e_{\eta'} = -e_\xi \sin \theta_o + e_\eta \cos \theta_o \tag{407}$$

and with

$$e_\perp = \sqrt{e_{\eta'}^2 + e_\zeta^2} \tag{408}$$

it follows that

$$\sin \phi_o = \frac{e_\zeta}{e_\perp} \tag{409}$$

$$\cos \phi_o = \frac{e_{\eta'}}{e_\perp} \tag{410}$$

From Fig. 23,

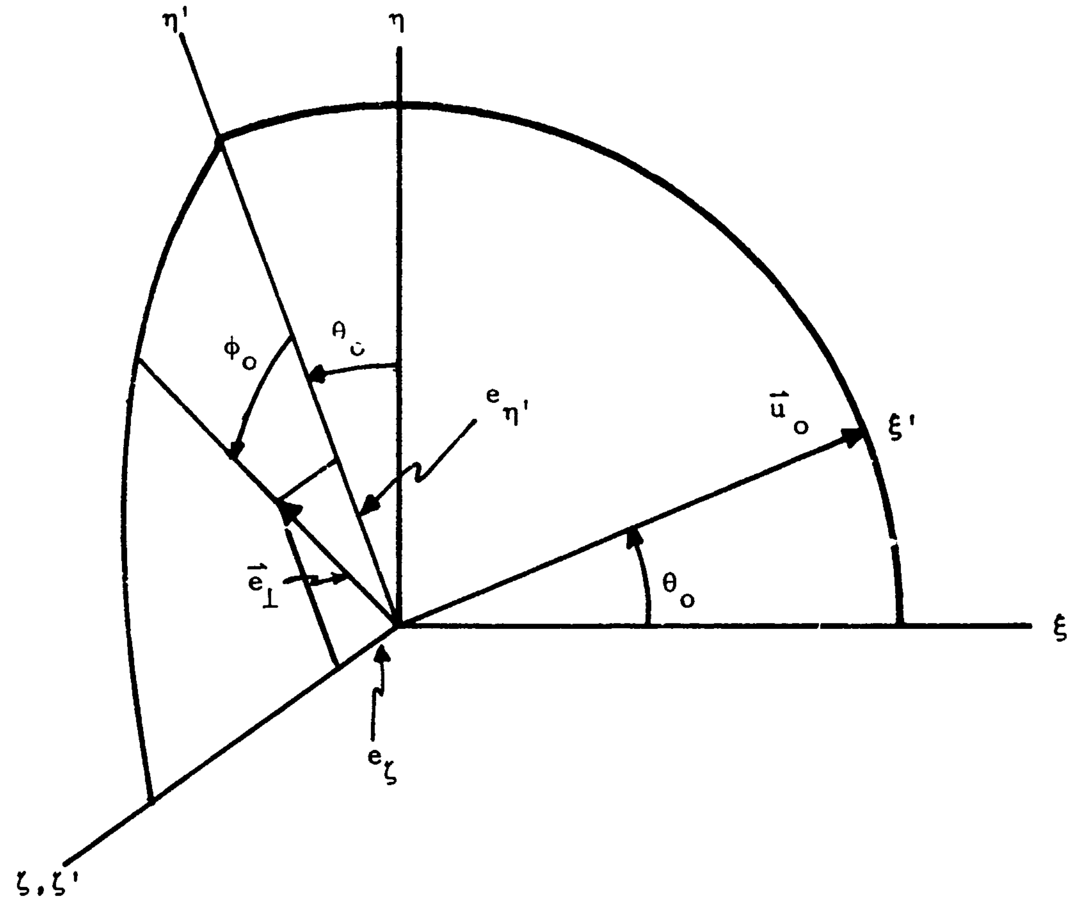


Figure 23
Geometry Defining ϕ_o

$$\begin{pmatrix} \vec{\xi}' \\ \vec{\eta}' \\ \vec{\zeta}' \end{pmatrix} = \begin{pmatrix} \cos\theta_o\ , & \sin\theta_o\ , & 0 \\ -\sin\theta_o\ , & \cos\theta_o\ , & 0 \\ 0\ , & 0\ , & 1 \end{pmatrix} \begin{pmatrix} \vec{\xi} \\ \vec{\eta} \\ \vec{\zeta} \end{pmatrix} \tag{411}$$

At firing time, directions 1, J, and K of Fig. 11 coincide with the directions of ξ', η', and ζ' of Fig. 23. Directions A, B, and 3 are related to the directions of ξ', η', and ζ' through the matrix relation

$$\begin{pmatrix} \vec{1}_A \\ \vec{1}_B \\ \vec{1}_3 \end{pmatrix} = \begin{pmatrix} \cos\delta_o\ , & \sin\delta_o\cos\phi_o\ , & \sin\delta_o\sin\phi_o \\ -\sin\delta_o\ , & \cos\delta_o\cos\phi_o\ , & \cos\delta_o\sin\phi_o \\ 0\ , & -\sin\phi_o\ , & \cos\phi_o \end{pmatrix} \begin{pmatrix} \vec{\xi}' \\ \vec{\eta}' \\ \vec{\zeta}' \end{pmatrix} \tag{412}$$

as can be seen from Figs. 11 and 23. If the components of $\vec{\omega}_o$ in the x_I, y_I, z_I coordinate system are ω_x, ω_y, and ω_z, it follows from Eqs. (395) and (411) that

$$\begin{pmatrix} \omega_{\xi'} \\ \omega_{\eta'} \\ \omega_{\zeta'} \end{pmatrix} = \begin{pmatrix} \cos\theta_o\sin B'\ , & \cos\theta_o\cos B'\ , & \sin\theta_o \\ -\sin\theta_o\sin B'\ , & -\sin\theta_o\cos B'\ , & \cos\theta_o \\ \cos B'\ , & -\sin B'\ , & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{413}$$

and from Eq. (412)

$$
\begin{pmatrix} \omega_{Ao} \\ \omega_{Bo} \\ \omega_{3o} \end{pmatrix} = \begin{pmatrix} \cos\delta_o , & \sin\delta_o \cos\phi_o , & \sin\delta_o \sin\phi_o \\ -\sin\delta_o & , \cos\delta_o \cos\phi_o , & \cos\delta_o \sin\phi_o \\ 0 , & -\sin\phi_o , & \cos\phi_o \end{pmatrix} \begin{pmatrix} \omega_{\xi'} \\ \omega_{\eta'} \\ \omega_{\zeta'} \end{pmatrix} \tag{414}
$$

This completes the set of initial conditions for Euler angle formulation.

4.3 Initial Conditions for the Approximate Equations - Initial conditions for the approximate equations of Section IV, Subsection 4 are developed in the same manner as those for the preceding subsection. New parameters are

$$P = 0$$
$$\dot{P} = u_o$$
$$S_2 = 0$$
$$\dot{S}_2 = 0$$
$$S_3 = 0$$
$$\dot{S}_3 = 0$$

Parameters u_o, δ_o, ϕ_o, θ_o, and B' are calculated as before.

4.4 Siacci Calculations - No integration is required when the Siacci equations are used. However u_o, θ_o, B', and ϕ_o are needed and they are calculated as in Subsection 4.2.

SECTION VII

FURTHER COMMENTS

Whereas this report covers most of the aspects of exterior ballistics which are of interest in airborne applications, some items are excluded because they are adequately covered elsewhere. Subjects omitted for this reason include numerical integration, and the standard atmosphere.

Methods of numerical integration are required since it is believed to be impossible to obtain a closed-form solution to the projectile equations of motion. Techniques for numerical integration are well known, and can be found in many of the standard references such as Ref. 68. Alternately, specialized methods are applicable and one such method may be found in Ref. 69.

Since all trajectory calculations involve the air density and the speed of sound vs altitude, a means of calculating these quantities is needed. This need is met by Ref. 70 and a convenient model for computer application can be found in Ref. 58.

This report is chiefly concerned with the application of more or less standard methods to airborne fire control. New methods applicable to treating windage jump may be found in Section IV, however, and in Refs. 71 and 72, which contain a simplified set of approximate equations adaptable to onboard utilization. Methods developed in this report and in Refs. 71 and 72 essentially comprise the current state of the art in exterior ballistics for airborne applications. Models developed to date are applicable to the 20-mm, M56 round. Development of new methods for treating windage jump may be needed for new rounds, such as the new 20-mm, 25-mm, and 30-mm rounds under development.

APPENDIX

MATRIX NOTATION

The matrix notation explained here is useful in defining the coordinate transformations and equations which are necessary for setting up initial conditions for trajectory computations onboard an aircraft.

In Fig. 24, let S denote a right-handed, rectangular coordinate system with origin O and coordinate axes x, y, z. A point in space, as measured in S, has coordinates x, y, and z and can be represented as a column matrix

$$a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or as a vector

$$\vec{a} = \vec{e}_x x + \vec{e}_y y + \vec{e}_z z$$

where $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ are unit vectors in the indicated directions. If

$$e = (\vec{e}_x, \vec{e}_y, \vec{e}_z)$$

defines a row matrix, then matrix multiplication provides a connection between the matrix and vector notation for a point in S:

$$\vec{a} = e\,a \tag{415}$$

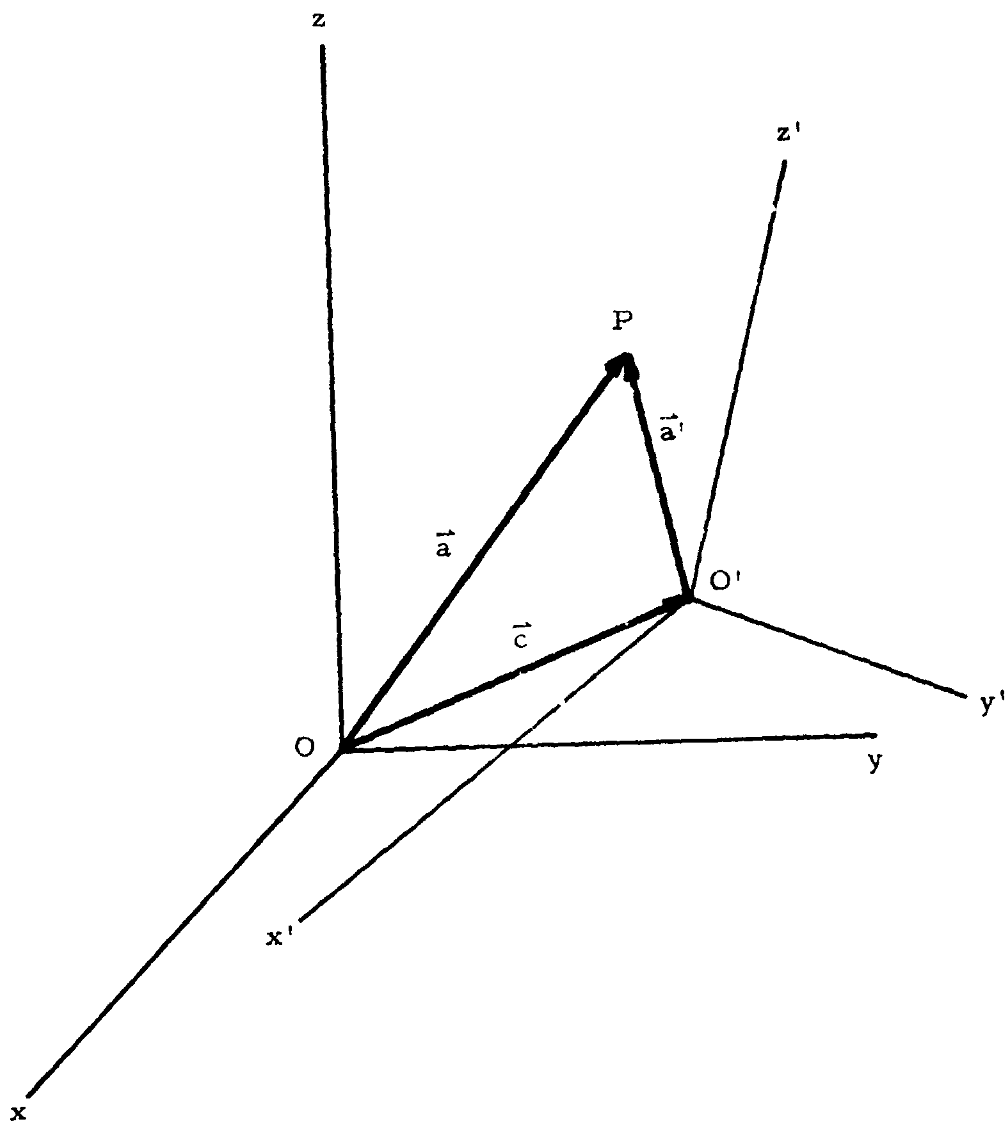


Figure 24
Relation between the S and S' Coordinate Systems

In Fig. 24, let S' denote another right-handed, rectangular coordinate system with origin O' and coordinate axes x', y', z'. If the matrix c, or the vector $\vec{c}$ = ec, denotes the position of O' as measured in S, the transformation of the coordinates of a point P, represented by a' in S', into the coordinates of P, represented by a in S, is given by the relation

$$a = c + T a' \tag{416}$$

T is a 3 × 3 matrix defining the angular orientation of the S' system with respect to the S system. If e' is the row matrix of unit vectors defining the x', y', z' directions in S', that is, if

$$e' = (\vec{e}_{x'}, \vec{e}_{y'}, \vec{e}_{z'})$$

then

$$e' = eT \tag{417}$$

and also

$$\tilde{e} = T \tilde{e}' \tag{418}$$

where the tilde (~) over a symbol denoting a matrix represents the transpose of that matrix.

It is a known property of an orthogonal transformation matrix, T, that the inverse, T^{-1}, equals the transpose, $\tilde{T}$. That is

$$\tilde{T} = T^{-1}$$

or

$$\tilde{T} T = I \tag{419}$$

where I is the unit matrix. It follows from Eq. (416) that

$$a' = \tilde{T}(a - c)$$

The relations above define transformations of position coordinates between S and S'. Equations for velocity transformations can be obtained by differentiation with respect to time. Differentiation of Eq. (419), where a dot denotes time differentiation, yields the result

$$\dot{\tilde{T}}\,T + \tilde{T}\,\dot{T} = 0 \qquad (420)$$

With

$$\Omega' = \tilde{T}\,\dot{T} \qquad (421)$$

it is seen that $\tilde{\Omega}' = \dot{\tilde{T}}\,T$ and so Eq. (420) may be written as

$$\Omega' + \tilde{\Omega}' = 0$$

It follows that the principal diagonal of Ω' is zero and $\Omega'_{ij} = -\Omega'_{ji}$, i.e., Ω' is skew-symmetric, and can be written in the form

$$\Omega' = \begin{pmatrix} 0 & , -\Omega_{z'} & , \Omega_{y'} \\ \Omega_{z'} & , 0 & , -\Omega'_{x'} \\ -\Omega_{y'} & , \Omega_{x'} & , 0 \end{pmatrix}$$

It can be shown that the components of the column matrix $\Omega' a'$ equal the corresponding components of the vector $\vec{\Omega}' \times \vec{a}'$. That is

$$e'\Omega' a' = \vec{\Omega}' \times \vec{a}'$$

where

$$\vec{\Omega}' = \vec{e}_{x'}\Omega_{x'} + \vec{e}_{y'}\Omega_{y'} + \vec{e}_{z'}\Omega_{z'}$$

Hence, $\vec{\Omega}'$ is identified as the angular velocity of S' relative to S measured in S'. It follows from Eq. (421) that

$$\dot{T} = T\Omega' \tag{422}$$

Differentiation of b = T b' yields

$$\dot{b} = T\dot{b}' + \dot{T}b'$$

where $\vec{b}$ = eb is a position vector. Hence, from Eq. (422)

$$\dot{b} = T\dot{b}' + T\Omega'b' = T(\dot{b}' + \Omega'b')$$

or, since e' = eT

$$e\dot{b} = e'\dot{b}' + e'\Omega'b'$$

This is equivalent to the familiar vector relation

$$\left(\frac{d\vec{b}}{dt}\right)_S = \left(\frac{d\vec{b}}{dt}\right)_{S'} + \vec{\Omega} \times \vec{b}$$

where subscripts S and S' denote the spaces in which the velocities are observed. If e' b' = $\vec{b}'$ represents a point P in S', then e' $\dot{b}'$ is the velocity of P as observed in S', e $\dot{b}$ is the velocity of P as observed in S, and e'Ω' b' is the component of velocity of P in S due to the angular motion of S'.

Note that if e is constant (i.e., if S is inertial), then from Eqs. (417) and (422)

$$\dot{e}' = e\dot{T} = eT\Omega' = e'\Omega'$$

If the column vector representation of Ω' in S' is

$$\omega' = \begin{pmatrix} \Omega_{x'} \\ \Omega_{y'} \\ \Omega_{z'} \end{pmatrix}$$

then

$$\omega = T\omega' \qquad (423)$$

is the column vector representing the components of Ω' in the S coordinate system. This follows from the fact that there are vectors $\vec{\Omega}$ and $\vec{b}$ in S such that $\vec{\Omega} \times \vec{b} = \vec{\Omega}' \times \vec{b}'$. Hence, $e\Omega b = e'\Omega' b' = (eT)\Omega' (\tilde{T} b) = e(T\Omega' \tilde{T})b$ and so $\Omega = T\Omega'\tilde{T}$. That Eq. (423) follows from $\Omega = T\Omega'\tilde{T}$ can be shown by comparison of matrix elements.

If U is the matrix representing a transformation from a space S" to S', that is, if

$$a' = U a''$$

and if T is the matrix representing a transformation from S' to S, that is, if

$$a = T a'$$

then

$$a = T U a'' = W a''$$

and

$$W = TU$$

is the matrix representing a transformation from S" to S. Then

$$\dot{W} = \dot{T} U + T \dot{U} \tag{424}$$

From Eq. (422), we may write

$$\dot{W} = W \Omega_W$$

$$\dot{T} = T \Omega_T$$

and

$$\dot{U} = U \Omega_U$$

where components of Ω_W and Ω_U are written with respect to S" and components of T are written in S' coordinates. Substitution of these expressions into Eq. (424) yields

$$W \Omega_W = T \Omega_T U + T U \Omega_U$$

or

$$W \Omega_W = T U \tilde{U} \Omega_T U + T U \Omega_U$$

Since $W = TU$ and $\tilde{W} = \tilde{U}\,\tilde{T}$, it follows that

$$U \Omega_W \tilde{U} = \Omega_T + U \Omega_U \tilde{U}$$

If ω_W, ω_T, and ω_U are the vector representations of Ω_W, Ω_T, and ω_U, respectively, then

$$U\omega_W = \omega_T + U\omega_U$$

This may be identified with the vector equations

$$\vec{\omega}_W = \vec{\omega}_T + \vec{\omega}_U$$

This development is a proof that angular velocities add, which is a result that is not intuitively obvious.

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13 ABSTRACT

Methods of exterior ballistics applicable to utilization in modern airborne fire control system design are documented herein. The fundamentals of exterior ballistics are included, along with a description of currently used ballistic and aerodynamic notations, a discussion of the limitations of the semi-empirical aerodynamic force and moment system, and methods of preparing aerodynamic data for use in trajectory computation. Tutorial material is provided to give the reader an understanding of windage jump caused by the complicated angular motion of a spinning projectile. The Siacci method is described and means for improving its accuracy are developed. Six-degree-of-freedom equations are derived in several different formulations for exploratory studies and digital computer computations, and the development of approximate equations for rapid evaluation of trajectory tables is included. Methods for calculating trajectory initial conditions are provided for shell fired from a turreted, gatling gun in a maneuvering aircraft, and the problems of ballistic and kinematic prediction are discussed briefly. The material covered herein should provide personnel in the Air Force and in industry with sufficient knowledge of exterior ballistics for advanced fire control system design.

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